Infinite Sums

A Project Work Submitted By
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Abstract

In this work, we explore almost all parts of the basic concepts in the infinite sums in the subject real analysis. Firstly, we begin our studies with finite sums, infinite unordered sums, ordered sums of series, and a criterion for convergence. There we build some basic constructions of convergent series then immediately we start to check series which is convergent by some simple tricks. Secondly, we study fourteen tests for convergence. We demonstrate some applications toward more generalized tests through examples where simpler tests fail to yield results. While our main focus of this project work on advanced tests for convergence, we also illustrate connections between the different tests. Thirdly, we then work on rearrangement, products of series, and summability method. At the end, we summarize more on infinite sums and infinite products.

Keywords: Finite sums, infinite sums, convergence, and series.

Declaration

I hereby declare that the work presented in this project work, submitted to Mathematical Association of Cambodia (MAC) in partial fulfillment of the requirements for being a member of MAC. I confirm that this project work is my original work, and that any work done by others or by myself previously has been acknowledged and referenced accordingly. This work was not previously presented to another examination board and has not been published.

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Introduction

In mathematics, a series is, roughly speaking, a description of the operation of adding infinitely many quantities, one after the other, to a given starting quantity. The study of series is a major part of calculus and its generalization, mathematical analysis. Series are used in most areas of mathematics, even for studying finite structures (such as in combinatorics) through generating functions. In addition to their ubiquity in mathematics, infinite series are also widely used in other quantitative disciplines such as physics, computer science, statistics and finance.

For a long time, the idea that such a potentially infinite summation could produce a finite result was considered paradoxical. This paradox was resolved using the concept of a limit during the 17th century. Zeno's paradox of Achilles and the tortoise illustrates this counterintuitive property of infinite sums: Achilles runs after a tortoise, but when he reaches the position of the tortoise at the beginning of the race, the tortoise has reached a second position; when he reaches this second position, the tortoise is at a third position, and so on. Zeno concluded that Achilles could never reach the tortoise, and thus that movement does not exist. Zeno divided the race into infinitely many sub-races, each requiring a finite amount of time, so that the total time for Achilles to catch the tortoise is given by a series. The resolution of the paradox is that, although the series has an infinite number of terms, it has a finite sum, which gives the time necessary for Achilles to catch up with the tortoise.

1.1 Brief Literature Review

Development of infinite series

Greek mathematician Archimedes produced the first known summation of an infinite series with a method that is still used in the area of calculus today. He used the method of exhaustion to calculate the area under the arc of a parabola with the summation of an infinite series, and gave a remarkably accurate approximation of π.

In the 17th century, James Gregory worked in the new decimal system on infinite series and published several Maclaurin series. In 1715, a general method for constructing the Taylor series for all functions for which they exist was provided by Brook Taylor. Leonhard Euler in the 18th century, developed the theory of hypergeometric series and q-series.

Convergence Criteria

The investigation of the validity of infinite series is considered to begin with Gauss in the 19th century. Euler had already considered the hypergeometric series

$$1 + \frac{\alpha\beta}{1 \cdot \gamma}x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1 \cdot 2 \cdot 2 \cdot \gamma(\gamma+1)}x^2 + \cdots$$

on which Gauss published a memoir in 1812. It established simpler criteria of convergence, and the questions of remainders and the range of convergence.

Cauchy (1821) insisted on strict tests of convergence; he showed that if two series are convergent their product is not necessarily so, and with him begins the discovery of effective criteria. The terms convergence and divergence had been introduced long before by Gregory (1668). Leonhard Euler and Gauss had given various criteria, and Colin Maclaurin had anticipated some of Cauchy's discoveries. Cauchy advanced the theory of power series by his expansion of a complex function in such a form.

Abel (1826) in his memoir on the binomial series

$$1 + \frac{m}{1!}x + \frac{m(m-1)}{2!}x^2 + ...$$

corrected certain of Cauchy's conclusions, and gave a completely scientific summation of the series for complex values of m and x. He showed the necessity of considering the subject of continuity in questions of convergence.

Cauchy's methods led to special rather than general criteria, and the same may be said of Raabe (1832), who made the first elaborate investigation of the subject, of De Morgan (from 1842), whose logarithmic test DuBois-Reymond (1873) and Pringsheim (1889) have shown to fail within a certain region; of Bertrand (1842), Bonnet (1843), Malmsten (1846, 1847, the latter without integration); Stokes (1847), Paucker (1852), Chebyshev (1852), and Arndt (1853).

General criteria began with Kummer (1835), and have been studied by Eisenstein (1847), Weierstrass in his various contributions to the theory of functions, Dini (1867), DuBois-Reymond (1873), and many others. Pringsheim's memoirs (1889) present the most complete general theory.

`https://en.wikipedia.org/wiki/Series_(mathematics)#:~:text=If`

1.2 Objectives of the Study

The objectives of this thesis work are as follows:

- Section 1,2,3,4,5: We introduce finite sums, infinite unordered sums, ordered sums, and criterion for convergence.
- Section 6: We study various tests for convergence respected to the different form of infinite sums.
- Section 7: We start introducing unconditional convergence, conditional convergence, and comparison of $\sum_{i=1}^{\infty} a_i$ and $\sum_{i\in\mathbb{N}} a_i$.
- Section 8: We start with products of series where products of absolutely convergent series and products of nonabsolutely convergent series. We study in a rigorous way.
- Section 9,10,11: We include summability method (Cesaro's and Abel's), more on infinite sums and infinite products.

1.3 Outline of the Project Work

In section 1, we start introducing finite sums, infinite unordered sums, ordered sums, and criterion for convergence. Furthermore, in section 2, we study tests for convergence. Moreover, we start introducing unconditional convergence, conditional convergence, and comparison of $\sum_{i=1}^{\infty} a_i$ and $\sum_{i\in\mathbb{N}} a_i$. After that, in section 4, we start with products of series that detail about products of absolutely convergent series and products of nonabsolutely convergent series. Finally, we include summability method (Cesaro's and Abel's), more on infinite sums and infinite products.

1.4 Preliminaries

To achieve our main objectives we have set for our work, we first have to build up all the basic concepts needed then immediately we go ahead with our main theorems which are followed by some useful applications, for each work.

To study deeply on infinite sums, we have to start from finite sums, cauchy criterion, and properties of series. Then we have to work on criterion for convergence to continue testing the convergence of infinite sums. After that we study rearrangement where we talk about uncondition and condition of convergence. Moreover, we continue our work on products of series, summability method, infinite products.

2
INFINITE SUMS

2.1 Introduction

Literally, infinite summations appears to have been studied and used long before any development of sequences and sequence limits. Indeed, even to form the notion of an infinite sum, it would seem that we should already have some concept of sequences, but this is not the way things developed. In was only by the time of Cauchy that the modern theory of infinite summation was developed using sequence limits as a basis for the theory. We can transfer a great deal of our expertise in sequential limits to the problem of infinite sums. Even so, in this work will show its own character and charm. In many ways infinite sums are much more interesting and important to analysis than sequences.

2.2 Finite Sums

There are a number of notation and a number of skills that we shall need to develop in order to succeed at the study of infinite sums that is to come. The notations of such summations may be novel. How best to write out a symbol indicating that some set of numbers

$$\{a_1, a_2, a_3, ..., a_n\}$$

has been summed? Certainly

$$a_1 + a_2 + a_3 + \cdots + a_n$$

is too cumbersome a way of writing such sums. The following have proved to communication much better

$$\sum_{i \in I} a_i$$

where I is the set $\{1, 2, 3, ..., n\}$ or

$$\sum_{1 \leq i \leq n} a_i \quad \text{or} \quad \sum_{i=1}^{n} a_i.$$

Here the Greek letter $\sum$, corresponding to an uppercase "S", is using to indicate a sum. It is to Leonhard Euler $(1707 - 1783)$ that we owe this sigma notation for sums (first used by him in 1755).

The usual rules of elementary arithmetic apply to finite sums. The commutative, associative, and distributive rules assume a different look when written in Euler's notation:

$$\sum_{i \in I} a_i + \sum_{i \in I} b_i = \sum_{i \in I} (a_i + b_i)$$

$$\sum_{i \in I} c a_i = c \sum_{i \in I} a_i$$

and

$$\left(\sum_{i\in I} a_i\right) \times \left(\sum_{i\in J} b_j\right) = \sum_{i\in I}\left(\sum_{j\in J} a_i b_j\right) = \sum_{j\in J}\left(\sum_{i\in I} a_i b_j\right).$$

Each of these can be checked mainly by determining the meanings and seeing that the notation produces the correct result.

Occasionally, in application of these ideas, one would like a simplified expression for a summation. The best known example is

$$\sum_{k=1}^{n} k = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2},$$

which is easily proved. When a sum of n terms for a general n has a simpler expression such as this it is usual to say that it has been expressed in closed form. Novices, seeing this, usually assume that any summation with some degree of regularity should allow a closed form expression. If not, what can we do with a sum that cannot be simplified?

One of the simplest of sums

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} = \sum_{k=1}^{n} \frac{1}{k}$$

does not allow any convenient formula, expressing the sum as some simple function of n. This is typical. It is only the rarest of summations that will allow simple formulas.

Even so, there are a few special cases that should be remembered and which make our task in some cases much easier.

Telescoping Sums. If a sum can be rewritten in the special form below, a simple computation (canceling s_1, s_2, etc.) gives the following closed form:

$$(s_1 - s_0) + (s_2 - s_1) + (s_3 - s_2) + (s_4 - s_3) + \cdots + (s_n - s_{n+1}) = s_n - s_0.$$

Example 2.2.1. *For a specific example of a sum that can be handled by considering it as telescoping, consider the sum*

$$\sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \frac{1}{4\cdot 5} + \cdots + \frac{1}{(n-1)\cdot n}.$$

A closed form is available since, using partial fractions, each term can be expressed as

$$\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}.$$

Thus

$$\sum_{k=1}^{n} \frac{1}{k(k+1)} = \sum_{k=1}^{n} \left(\frac{1}{k} - \frac{1}{k-1}\right) = 1 - \frac{1}{n+1}.$$

The exercises certain a number of other examples of this type.

Geometric Progressions. If the terms of a sum are in a geometric progression (i.e., if each term is some constant factor times the previous terms), then a closed form for any such sum is available:

$$1 + r + r^2 + \cdots + r^{n-1} + r^n = \frac{1 - r^{n+1}}{1 - r}. \qquad (1)$$

This assumes that $r \neq 1$; if $r = 1$ the sum is easily seen to be just $n + 1$. The formula in (1) can be proved by converting to a telescoping sum. Consider instead $(1 - r)$ times the preceding sum:

$$(1 + r)(1 + r + r^2 + \cdots + r^{n-1} + r^n) = (1 - r) + (r - r^2) + \cdots + (r^n - r^{n+1}).$$

Now add this up as a telescoping sum to obtain the formula stated in (1).

Any geometric progression assumes the form

$$A + Ar + Ar^2 + \cdots + Ar^n = A(1 + r + r^2 + \cdots + r^n)$$

and formula (1) is then applied.

Summation By Parts. Sums are frequently given in a form such as

$$\sum_{k=1}^{n} a_k b_k$$

for sequences $\{a_k\}$ and $\{b_k\}$. If a formula happens to be available for

$$s_n = a_1 + a_2 + \cdots + a_n,$$

then there is a frequently useful way of rewriting this sum (using $s_0 = 0$ for convenience):

$$\sum_{k=1}^{n} a_k b_k = \sum_{k=1}^{n} (s_k - s_{k-1}) b_k$$

$$= s_1(b_1 - b_2) + s_2(b_2 - b_3) + \cdots + s_{n-1}(b_{n-1} - b_n) + s_n b_n.$$

2.3 Infinite Unordered sums

We now pass to the study of infinite sums. We wish to interpret

$$\sum_{i \in I} a_i$$

for an index set I that is infinite.

To begin our study imagine that we are given a collection of numbers a_i indexed over an infinite set I (i.e., there is a function $a : I \longrightarrow \mathbb{R}$) and we wish the sum of the totality of these numbers. If the set I has some structure, then we can use that structure to decide how to start adding the numbers. For example, if a is a sequence so that $I = \mathbb{N}$, then we should likely start adding at the beginning of the sequence:

$$a_1, a_1 + a_2, a_1 + a_2 + a_3, a_1 + a_2 + a_3 + a_4, \ldots$$

and so defining the sum as the limit of this sequence of partial sums.

Another set I would suggest a different order. For example, if $I = \mathbb{Z}$, then a popular method of adding these up would be to start off:

$$a_0, a_{-1} + a_0 + a_1$$

$$a_{-2} + a_{-1} + a_0 + a_1 + a_2$$

$$a_{-3} + a_{-2} + a_{-1} + a_0 + a_1 + a_2 + a_3, ...$$

once again defining the sum as the limit of this sequence.

It seems that the method of summation and hence defining the meaning of the expression

$$\sum_{i \in I} a_i$$

for infinite sets I must depend on the nature of the set I and hence on the particular problems of the subject one is studying. This is true to some extent. But it does not stop us from inventing a method that will apply to all infinite sets I. We must make a definition that takes account of no extra structure or ordering for the set I and just treats it as a set. This is called the unordered sum and the notation $\sum_{i \in I} a_i$ is always meant to indicate that an unordered sum is being considered. The key is just how to pass from finite sums to infinite sums. Both of the previous examples used the idea of taking some finite sums and then passing to a limit.

Definition 2.3.1. *[1] Let I be an infinite set and a function $a : I \to \mathbb{R}$. Then we write*

$$\sum_{i \in I} a_i = c$$

and say that the sum converges if for every $\epsilon > 0$ there is a finite set $I_0 \subset I$ so that, for every finite set $J, I_0 \subset J \subset I$,

$$\left| \sum_{i \in J} a_i \right| < \epsilon.$$

A sum that does not converge is said to diverge.

Note that we never form a sum of infinitely many terms. The definition always computes finite sums.

Example 2.3.2. *Let us show, directly from the definition, that*

$$\sum_{i \in \mathbb{Z}} 2^{|-i|} = 3.$$

If we first sum

$$\sum_{-N \leq i \leq N} 2^{|-i|}$$

by rearranging the terms into the sum

$$1 + 2(2^{-1} + 2^{-2} + ... + 2^{-N})$$

we can see why the sum is likely to be 3. Let $\epsilon > 0$ and choose N so that $2^{-N} < \epsilon/4$. From the formula for a finite geometric progression we have

$$\left| \sum_{-N \leq i \leq N} 2^{-|i|} - 3 \right| = 2 \left| (2^{-1} + 2^{-2} + ... + 2^{-N}) - 1 \right| < 2(2^{-N}) < \epsilon/2.$$

Also, if $J \subset \mathbb{Z}$ with J finite geometric progression, then

$$\sum_{j \in J} 2^{-|j|} < 2(2^{-N}) < \epsilon/2$$

again from the formula for a finite geometric progression. Let

$$I_0 = \{i \in \mathbb{Z} : -N \leq i \leq N\}.$$

If $I_0 \subset J \subset \mathbb{Z}$ with J finite then

$$\left| \sum_{i \in J} 2^{-|i|} - 3 \right| = \left| \sum_{-N \leq i \leq N} 2^{-|i|} - 3 \right| + \sum_{i \in J \setminus I_0} 2^{-|i|} < \epsilon$$

as required.

2.3.1 Cauchy Criterion

In most theories of convergence one asks for a necessary and sufficient condition for convergence. Here is the Cauchy criterion for sums.

Theorem 2.3.3. *[1] A necessary and sufficient condition that the sum $\sum_{i \in I} a_i$ converges is that for every $\epsilon > 0$ there is a finite set I_0 so that*

$$\left| \sum_{i \in J} a_i \right| < \epsilon$$

for every finite set $J \subset I$ that contains no element of I_0 (i.e., for all finite sets $J \subset I \setminus I_0$).

Proof. As usual in Cauchy criterion proofs, one direction is easy to prove.
Suppose that $\sum_{i \in I} a_i = C$ converges. Then for every $\epsilon > 0$ there is a finite set I_0 so that

$$\left| \sum_{i \in K} a_i - C \right| < \epsilon/2$$

for every finite set $I_0 \subset K \subset I$. Let $J \subset I \setminus I_0$ and consider taking a sum over $K = I_0 \cup J$. Then

$$\left| \sum_{i \in I_0 \cup J} -C \right| < \epsilon/2$$

and

$$\left| \sum_{i \in I_0} -C \right| < \epsilon/2.$$

By subtracting these two inequalities and remembering that

$$\sum_{i \in I_0 \cup J} a_i = \sum_{i \in J} a_i + \sum_{i \in I_0} a_i$$

(since I_0 and J are disjoint) we obtain

$$\left|\sum_{i \in J} a_i\right| < \epsilon.$$

This is exactly the Cauchy criterion.

Conversely, suppose that the sum does satisfy the Cauchy criterion. Then, applying that criterion to $\epsilon = 1, 1/2, 1/3, ...$ we can choose a sequence of finite sets $\{I_n\}$ so that

$$\left|\sum_{i \in J} a_i\right| < 1/n$$

for every finite set $J \subset I \backslash I_n$. We can arrange our choices to make

$$I_1 \subset I_2 \subset I_3 \subset ...$$

so that the sequence of sets is increasing.
Let

$$c_n = \sum_{i \in I_n} a_i$$

Then for any $m > n$,

$$|c_n - c_m| = \left|\sum_{i \in I_m \backslash I_n} a_i\right| < 1/n.$$

It follows from this that $\{c_n\}$ is a Cauchy sequence of real numbers and hence converges to some real number c. Let $\epsilon > 0$ and choose N so that $N > 2/\epsilon$. Then, for any $n > N$ and any finite set J with $I_N \subset J \subset I$,

$$\left|\sum_{i \in J} a_i - c\right| \leq \left|\sum_{i \in I_N} a_i - c_N\right| + |c_N - c| + \left|\sum_{i \in J \backslash I_n} a_i\right| < 0 + 2/N < \epsilon.$$

By definition, then,

$$\sum_{i \in I_N} a_i = c$$

and the theorem is proved. □

All But Countably Many Terms in a Convergent Sum Are Nonzero. Our next theorem shows that having "too many" numbers to add up causes problems. If the set I is not countable then most of the a_i that we are to add up should be zero if the sum is to exist. This shows too that theory of sums is in an essential way limited to taking sums over countable sets. It is notationally possible to have a sum

$$\sum_{x \in [0,1]} f(x)$$

but that sum cannot be defined unless $f(x)$ is mostly zero with only countably many exceptions.

Theorem 2.3.4. *Suppose that* $\sum_{i \in I} a_i$ *converges. Then* $a_i = 0$ *for* $a_i = 0$ *for all* $i \in I$ *except for a countable subset of* I.

Proof. We know for any convergent sum there is a positive integer M so that all the sums

$$\left|\sum_{i\in I_0} a_i\right| \leq M$$

for any finite set $I_0 \subset I$. Let m be an integer. We ask how many elements a_i are there such that $a_i > 1/m$? It is easy to see that there are at most Mm of them since if there oure any more our sum would exceed M. Similarly, there are at most Mm terms such that $-a_i > 1/m$. Thus each element of $\{a_i : i \in I\}$ that is not zero can be given a "rank" m depending on whether

$$1/m < a_i \leq 1/(m-1) \quad \text{or} \quad -a_i \leq 1/(m-1).$$

As there are only finitely many elements at each rank, thin gives us a method for listing all of the nonzero elements in $\{a_i : i \in I\}$ and so this set is countable. □

2.4 Ordered Sums: Series

For the vast majority of applications, one wished to sum not an arbitrary collection of numbers but most commonly some sequence of numbers:

$$a_1 + a_2 + a_3 + ...$$

The set $\mathbb{N}$ of natural numbers has an order structure, and it is not in our best interests to ignore that order since that is the order in which the sequence is presented to us.

The most compelling way to add up a sequence of numbers is to begin accumulating:

$$a_1, a_1 + a_2, a_1 + a_2 + a_3, a_1 + a_2 + a_3 + a_4, ...$$

and to define the sum as the limit of this sequence. This is what we shall do.

If we studied Section 3.3 on unordered summation we should also compare this "ordered" method with the unordered method. The ordered sum of a sequence is called a series and the notation

$$\sum_{k=1}^{\infty} a_k$$

is used exclusively for this notion.

Definition 2.4.1. *Let $\{a_k\}$ be a sequence of real numbers. Then we write*

$$\sum_{k=1}^{\infty} a_k = c$$

and say that the series converges if the sequence

$$s_n = \sum_{k=1}^{n} a_k$$

(called the sequence of partial sums of the series) converge to c. If the series does not converge it is said to be divergent.

The definition reduces the study of series to the study of sequence.

2.4.1 Properties

[5]The following short harvest of theorems we obtain directly from our sequence theory. The convergence or divergence of a series $\sum_{k=1}^{\infty} a_k$ depends on the convergence or divergence of the sequence of partial sums

$$s_n = \sum_{k=1}^{n} a_k$$

and the value of the series is the limit of the sequence.

Theorem 2.4.2. *If a series $\sum_{k=1}^{\infty} a_k$ converges, then the sum is unique.*

Theorem 2.4.3. *If both series $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ converge, then so too does the series*

$$\sum_{k=1}^{\infty}(a_k + b_k)$$

and

$$\sum_{k=1}^{\infty}(a_k + b_k) = \sum_{k=1}^{\infty} a_k + \sum_{k=1}^{\infty} b_k.$$

Theorem 2.4.4. *If the series $\sum_{k=1}^{\infty} a_k$ converges, then so too does the series $\sum_{k=1}^{\infty} ca_k$ for any real number c and*

$$\sum_{k=1}^{\infty} ca_k = c\sum_{k=1}^{\infty} a_k.$$

Theorem 2.4.5. *If both series $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ converge and $a_k \leq b_k$ for each k, then*

$$\sum_{k=1}^{\infty} a_k \leq \sum_{k=1}^{\infty} b_k.$$

Theorem 2.4.6. *Let $M \geq 1$ be any integer. Then the series*

$$\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + a_4 + ...$$

converges if and only if the series

$$\sum_{k=1}^{\infty} a_{M+k} = a_{M+1} + a_{M+2} + a_{M+3} + a_{M+4} + ...$$

converges.

Note. If we call $\sum_{p}^{\infty} a_i$ a "tail" for the series $\sum_{1}^{\infty} a_i$, then we can say that this last theorem asserts that it is the behavior of the tail that determines the convergence or divergence of the series. Thus in questions of convergence we can easily ignore the first part of the series, however many terms we like. Naturally, the actual sum of the series will depend on having all the terms.

2.4.2 Special Series

Telescoping Series Any series for which we can find a closed form for the partial sums we should probably be able to handle by sequence methods. Telescoping series are the easiest to deal with.

If the sequence of partial sums of a series can be computed in some closed form $\{s_n\}$, then the series can be rewritten in the telescoping form

$$(s_1) + (s_2 - s_1) + (s_3 - s_2) + (s_4 - s_3) + \cdots + (s_n - s_{n-1}) \ldots$$

and the series studied by means of the sequence $\{s_n\}$.

Example 2.4.7. *Consider the series*

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = \sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+1} \right) = \lim_{n \longrightarrow \infty} \left(1 - \frac{1}{n+1} \right) = 1$$

with an easily computable sequence of partial sums.

Do not be too encouraged by the apparent ease of the method illustrated by the example. In practice we can hardly ever do anything but make a crude estimate on the size of the partial sums. An exact expression, as we have here, would be rarely available. Even so, it is entertaining and instructive to handle a number of series by such a method.

Geometric Series Geometric series from another convenient class of series that we can handle simply by sequence methods. From the elementary formula

$$1 + r + r^2 + \cdots + r^{n-1} + r^n = \frac{1 - r^{n+1}}{1 - r} \quad (r \neq 1)$$

we see immediately that the study of such a series reduces to the computation of the limit

$$\lim_{n \longrightarrow \infty} \frac{1 - r^{n+1}}{1 - r} = \frac{1}{1 - r}$$

which is valid for $-1 < r < 1$ (which is usually expressed as $|r| < 1$) and invalid for all other values of r. Thus, for $|r| < 1$ the series

$$\sum_{k=1}^{\infty} r^{k-1} = 1 + r + r^2 + \cdots = \frac{1}{1 - r} \qquad (2)$$

and is convergent and for $|r| \geq 1$ the series diverges. It is well worthwhile to memorize this fact and formula (2) for the sum of the series.

Harmonic Series As a first taste of an elementary looking series that presents a new challenge to our methods, consider the series

$$\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots ,$$

which is called the harmonic series. Let us show that this series diverge.

This series has no closed form for the sequence of partial sum $\{s_n\}$ and so there seems no hope of merely computing $\lim_{n\longrightarrow\infty} s_n$ to obtain convergence/divergence of the harmonic series. But we can make estimates on the size of s_n even if we cannot compute it directly. The sequence of partial sums increases at each step, and if we watch only at the step $1, 2, 4, 8, \ldots$ and make a rough loour estimate of $s_1, s_2, s_4, s_8, \ldots$ we see that $s_{2^n} \geq 1 + n/2$ for all n. From this we see that $\lim_{n\longrightarrow\infty} s_n = \infty$ and so the series diverges.

Alternating Harmonic Series A variant on the harmonic series presents immediately a new challenge. Consider the series

$$\sum_{k=1}^{\infty}(-1)^{k-1}\frac{1}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4}\cdots,$$

which is called the alternating harmonic series.

The reason why this presents a different challenge is that the sequence of partial sums is no longer increasing. Thus estimates as to how big that sequence get may be of no help. We can see that the sequence is bounded, but that does not imply convergence for a non monotonic sequence. Once again, we have no closed form for the partial sums so that a routine computation of a sequence limit is not available.

By computing the partial sums $s_2, s_4, s_6, \ldots$ we see that the subsequence $\{s_{2n}\}$ is increasing. By computing the partial sums $s_1, s_2, s_5, \ldots$ we see that the subsequence $\{s_{2n-1}\}$ is decreasing. A few more observations show us that

$$1/2 = s_2 < s_4 < s_6 < \cdots < s_5 < s_3 < s_1 = 1. \qquad (3)$$

Our theory of sequences now allows us to assert that both limits

$$\lim_{n\longrightarrow\infty} s_{2n} \quad \text{and} \quad \lim_{n\longrightarrow\infty} s_{2n-1}$$

exist. Finally, since

$$s_{2n} - s_{2n-1} = \frac{-1}{2n} \longrightarrow 0$$

we can conclude that $\lim_{n\longrightarrow\infty} s_n$ exists. It is somewhere between $\frac{1}{2}$ and 1 because of the inequalities (3) but exactly what it is would take much further analysis. Thus we have proved that the alternating harmonic series converges (which is in contrast to the divergence of the harmonic series).

***p*-harmonic series** The series

$$\sum_{k=1}^{\infty}\frac{1}{k^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots$$

for any parameter $0 < p < \infty$ is called the **p-harmonic series**. The methods we have used in the study of the harmonic series can be easily adapted to handle this series. As a first observation note that if $0 < p < 1$, then

$$\frac{1}{k^p} > \frac{1}{k}.$$

Thus the p-harmonic series for $0 < p < 1$ is large than the harmonic series, small enough it turns out that the series converges. To show this we can group the terms in the same manner as before for the harmonic series and obtain

$$1 + \left[\frac{1}{2^p} + \frac{1}{3^p}\right] + \left[\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p}\right] + \left[\frac{1}{8^p} + \cdots + \frac{1}{15^p}\right] + \cdots$$

$$\leq 1 + \frac{2}{2p} + \frac{4}{4^p} + \frac{8}{8^p} \leq \frac{1}{1 - 2^{1-p}}$$

since we recognize the letter series as a convergent geometric series with ratio 2^{1-p}. In this way we obtain an upper bound for the partial sums of the series

$$\sum_{k=1}^{\infty} \frac{1}{k^p}$$

for all $p > 1$. Since the partial sums are increasing and bounded above, the series must converge.

Size of the Terms It should seem apparent from the examples we have seen that a convergent series must have ultimately small terms. If $\sum_{k=1}^{\infty} a_k$ converges, then it seems that a_k must tend to 0 as k gets large. Certainly, for the geometric series that idea precisely described the situation:

$$\sum_{k=1}^{\infty} r^{k-1}$$

converges if $|r| < 1$, which is exactly when the terms tend to zero and diverges when $|r| \geq 1$, which is exactly when the terms do not tend to zero.

A reasonable conjecture might be that this is always the situation: A series $\sum_{k=1}^{\infty} a_k$ converges if and only if $a_k \longrightarrow 0$ as $k \longrightarrow \infty$. But we have already seen the harmonic series diverges even though its terms do get small; they simply don't get small fast enough. Thus the correct observation is simple and limited.

$$\text{If} \quad \sum_{k=1}^{\infty} a_k \quad \text{converges, then} \quad a_k \longrightarrow 0 \quad \text{as} \quad k \longrightarrow \infty.$$

To check this is easy. If $\{s_n\}$ is the sequence of partial sums of a convergent series $\sum_{k=1}^{\infty} a_k = C$, then

$$\lim_{n \longrightarrow \infty} a_n = \lim_{n \longrightarrow \infty} (s_n - s_{n-1}) = \lim_{n \longrightarrow \infty} s_n - \lim_{n \longrightarrow \infty} s_{n-1} = C - C = 0.$$

The converse, as we just noted, is false. To obtain convergence of the series it is not enough to know that the terms tend to zero. We shall see, though, that many of the tests that follow discuss the rate at which the terms tend to zero.

2.5 Criterion for Convergence

How do we determine the convergence or divergence of a series? The meaning of convergence or divergence is directly given in terms of the sequence of partial sums. But usually it is very difficult to say much about that sequence. Certainly we hardly ever get a closed form for the partial sums.

For a successful theory of series we need some criteria that will enable us to assert the convergence or divergence of a series without much bothering with an intimate acquaintance with the sequence of partial sums. The following material begins the development of these criteria.

2.5.1 Boundedness Criterion

If a series $\sum_{k=1}^{\infty} a_k$ consists entirely of nonnegative terms, then it is clear that the sequence of partial sums forms a monotonic sequence. It is strictly increasing if all terms are positive.

We have a well-established fundamental principle for the investigation of all monotonic sequences:

A monotonic sequence is convergent if and only if it is bounded.

Applied to the study of series, this principle says that a series $\sum_{k=1}^{\infty} a_k$ consisting entirely of nonnegative terms will converge if the sequence of partial sums is bounded and will diverge if the sequence of partial sums is unbounded.

This reduce the study of the convergence or divergence behavior of such series to inequality problems:
Is there or is there not a number M so that

$$s_n = \sum_{k=1}^{n} a_k \leq M$$

for all integers n?

This is both good news and bad. Theoretically it means that convergence problems for this special class of series reduce to another problem: one of boundedness. That is good news, reducing an apparently difficult problem to one we already understand. The bad news is that inequality problems may still be difficult.

Note. A word of warning. The boundedness of the partial sums of a series is not of as great an interest for series where the terms can be both positive and negative. For such series the boundedness of the partial sums does not guarantee convergence.

2.5.2 Cauchy Criterion

One of our main theoretical tools in the study of convergent sequences is the Cauchy criterion describing (albeit somewhat technically) a necessary and sufficient condition for a sequence to be convergent.

If we translate that criterion to the language of series we shall then have a necessary and sufficient condition for a series to be convergent. Again it is rather technical and mostly useful in developing a theory rather than in testing specific series. The translation is nearly immediate.

Definition 2.5.1. *The series*

$$\sum_{k=1}^{\infty} a_k$$

is said to satisfy the Cauchy criterion for convergence provided that for every $\epsilon > 0$ there is an integer N so that all of the finite sums

$$\left| \sum_{k=n}^{m} a_k \right| < \epsilon$$

for any $N \leq n < m < \infty$.

Now we have a principle that can be applied in many theoretical situations:

A series $\sum_{k=1}^{\infty} a_k$ converges if and only if it satisfies the Cauchy criterion for convergence.

Note. It may be useful to think of this conceptually. The criterion asserts that convergence is equivalent to the fact that blocks of terms

$$\sum_{k=N}^{M} a_k$$

added up and taken from far on in the series must be small. Loosely we might describe this by saying that a convergent series has a "small tail."

Note too that if the series converges, then this criterion implies that for every $\epsilon > 0$ there is an integer N so that

$$\left|\sum_{k=n}^{\infty} a_k\right| < \epsilon$$

for every $n \geq N$.

2.5.3 Absolute Convergence

If a series consists of nonnegative terms only, then we can obtain convergence or divergence by estimating the size of the partial sums. If the partial sums remain bounded, then the series converges, if not, the series diverges.

No such conclusion can be made for a series $\sum_{k=1}^{\infty} a_k$ of positive and negative numbers. Boundedness of the partial sums does not allow us to conclude anything about convergence or divergence since the sequence of partial sums would not be monotonic. What we can do is ask whether there is any relation between the two series

$$\sum_{k=1}^{\infty} a_k \quad \text{and} \quad \sum_{k=1}^{\infty} |a_k|$$

where the letter series has had the negative signs stripped from it. We shall see that convergence of the series of absolute values ensures convergence of the original series. Divergence of the series of absolute values gives, however no information.

This gives us a useful test that will prove the convergence of a series $\sum_{k=1}^{\infty} a_k$ by investigating instead the related series $\sum_{k=1}^{\infty} |a_k|$ without the negative signs.

Theorem 2.5.2. *If the series $\sum_{k=1}^{\infty} |a_k|$ converges, then so too does the series $\sum_{k=1}^{\infty} a_k$.*

Proof. The proof takes two applications of the Cauchy criterion. If $\sum_{k=1}^{\infty} |a_k|$ converges, then for every $\epsilon > 0$ there is an integer N so that all of the finite sums

$$\sum_{k=n}^{m} |a_k| < \epsilon$$

for any $N < n < m < \infty$. But then

$$\left|\sum_{k=n}^{m} a_k\right| \leq \sum_{k=n}^{m} |a_k| < \epsilon.$$

It follows, by the Cauchy criterion applied to the series $\sum_{k=1}^{\infty} a_k$, that this series is convergent. □

Note. Note that there is no claim in the statement of this theorem that the two series have the same sum, just that the convergence of one implies the convergence of the other.

For theoretical reasons it is important to know when the series $\sum_{k=1}^{\infty} |a_k|$ of absolute values converges. Such series are "more" than convergent. There are convergent in a way that allows more manipulations than would otherwise be available. They can be thought of as more robust; a series that converges, but whose absolute series does not converge is in some ways fragile. This leads to the following definitions.

Definition 2.5.3. *A series $\sum_{k=1}^{\infty} a_k$ is said to be absolutely convergent if the related series $\sum_{k=1}^{\infty} |a_k|$ converges.*

Definition 2.5.4. *A series $\sum_{k=1}^{\infty} |a_k|$ is said to be nonabsolutely convergent if the series $\sum_{k=1}^{\infty} a_k$ converges but the series $\sum_{k=1}^{\infty} |a_k|$ diverges.*

Note that every absolutely convergent series is also convergent. We think of it as "more than convergent." Fortunately, the terminology preserves the meaning even though the "absolutely" refers to the absolute value, not to any other implied meaning. This play no words would not be available on all languages.

Example 2.5.5. *Using this terminology, applied to series we have already studied, we can know assert the following:*
Any geometric series $1 + r + r^2 + r^3 + ...$ is absolutely convergent if $|r| < 1$ and divergent if $|r| \geq 1$. and the alternating harmonic series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4}...$ is non absolutely convergent.

2.6 Tests for Convergence

[4]In many investigations and applications of series it is important to recognize that a given series converges, converges absolutely, or diverges. Frequently the sum of the series is not of much interest, just the convergence behavior. Over the years a battery of tests have been developed to make this task easier.

There are only a few basic principles that we can use to check convergence or divergence and we have already discussed these in Section 2.5. One of the most basic is that a series of nonnegative terms is convergent if and only if the sequence of partial sums is bounded. Most of the tests in the sequel are just clever ways of checking that the partial sums are bounded without having to do the computations involved in finding that upper bound.

2.6.1 Trivial Test

The first test is just an observation that we have already made about series: If a series $\sum_{k=1}^{\infty} a_k$ converges, then $a_k \longrightarrow 0$. We turn this into a divergence test. For example, some novices will worry for a long time over a series such as

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt[k]{k}}$$

applying a battery of tests to it to determine convergence. The simplest way to see that this series diverges is to note that the terms tend to 1 as $k \longrightarrow \infty$. Perhaps this is the first thing that should be considered for any series. If the terms do not get small there is no point puzzling whether the series converges. It does not.

(Trivial Test) If the terms of the series $\sum_{k=1}^{\infty}$ do not converge to 0, then the series diverges.

Proof. We have already proved this, but let us prove it now as a special case of the Cauchy criterion. For all $\epsilon > 0$ there is an N so that

$$|a_n| = \left| \sum_{k=n}^{n} a_k \right| < \epsilon$$

for all $n \geq N$ and so, by definition, $a_k \longrightarrow 0$. □

2.6.2 Direct Comparison Tests

A series $\sum_{k=1}^{\infty} a_k$ with all terms nonnegative can be handled by estimating the size of the partial sums. Rather than making a direct estimate it is sometimes easier to find a bigger series that converges. This larger series provides an upper bound for our series without the need to compute one ourselves.

Note. Make sure to apply these tests only for series with nonnegative terms since, for arbitrary series, this information is useless.

(Direct Comparison Test I) Suppose that the terms of the series $\sum_{k=1}^{\infty} a_k$ are each smaller than the corresponding terms of the series $\sum_{k=1}^{\infty} b_k$; that is, that

$$0 \leq a_k \leq b_k$$

for all k. If the larger series converges, then so does the smaller series.

Proof. If $0 \leq a_k \leq b_k$ for all k, then

$$\sum_{k=1}^{n} a_k \leq \sum_{k=1}^{n} b_k \leq \sum_{k=1}^{\infty} b_k.$$

Thus the number $B = \sum_{k=1}^{\infty} b_k$ is an upper bound for the sequence of partial sums of the series $\sum_{k=1}^{\infty} a_k$. It follows that $\sum_{k=1}^{\infty} a_k$ must converge. □

Note. In applying this and subsequent tests that demand that all terms of a series satisfy some requirement, we should remember that convergence and divergence of a series $\sum_{k=1}^{\infty} a_k$ depends only on the behavior of a_k for large values of k. Thus this test (and many others) could be reformulated so as to apply only for k greater than some integer N.

(Direct Comparison Test II) Suppose that the terms of the series $\sum_{k=1}^{\infty} a_k$ are each larger than the corresponding terms of the series $\sum_{k=1}^{\infty} c_k$; that is, that

$$0 \leq c_k \leq a_k$$

for all k. If the smaller series diverges, then so does the larger series.

Proof. This follows from Test 3.19 since if the larger series did not diverge, then it must converge and so too must the smaller series. □

Here are two examples illustrating how these tests may be used.

Example 2.6.1. *Consider the series*

$$\sum_{k=1}^{\infty} \frac{k+5}{k^3+k^2+k+1}.$$

While the partial sums might seem hard to estimate at first, a fast glance suggests that the terms (crudely) are similar to $1/k^2$ for large values of k and we know that the series $\sum_{k=1}^{\infty} 1/k^2$ converges. Note that

$$\frac{k+5}{k^3+k^2+k+1} = \frac{1+5/k}{k^2(1+1/k+1/k^2+1/k^3)} \leq \frac{C}{k^2}$$

for some choice of C (e.g., $C = 6$ will work). We now claim that our given series converges by a direct comparison with the convergent series $\sum_{k=1}^{\infty} C/k^2$. (This is a p-harmonic series with $p = 2$.)

Example 2.6.2. *Consider the series*

$$\sum_{k=1}^{\infty} \sqrt{\frac{k+5}{k^2+k+1}}.$$

Again, a fast glance suggests that the terms (crudely) are similar to $1/\sqrt{k}$ for large values of k and we know that the series $\sum_{k=1}^{\infty} 1/\sqrt{k}$ diverges. Note that

$$\frac{k+5}{k^2+k+1} = \frac{1+5/k}{k(1+1/k+1/k^2)} \geq \frac{C}{k}$$

for some choice of C (e.g., $C = \frac{1}{4}$ will work). We now claim that our given series diverges by a direct comparison with the divergent series $\sum_{k=1}^{\infty} \sqrt{C}/\sqrt{k}$. (This is a p-harmonic series with $p = 1/2$.)

The examples show both advantages and disadvantages to the method. We must invent the series that is to be compared and we must do some amount of inequality work to show that comparison. The next tests replace the inequality work with a limit operation, which is occasionally easier to perform.

2.6.3 Limit Comparison Tests

We have seen that a series $\sum_{k=1}^{\infty} a_k$ with all terms nonnegative can be handled by comparing with a larger convergent series or a smaller divergent series. Rather than check all the terms of the two series being compared, it is convenient sometimes to have this checked automatically by the computation of a limit. In this section, since the tests involve a fraction, we must be sure not only that all terms are nonnegative, but also that we have not divided by zero.

(Limit Comparison Test I) Let each $a_k \geq 0$ and $b_k > 0$. If the terms of the series $\sum_{k=1}^{\infty} a_k$ can be compared to the terms of the series $\sum_{k=1}^{\infty} b_k$ by computing

$$\lim_{k \longrightarrow \infty} \frac{a_k}{b_k} < \infty$$

and if the latter series converges, then so does the former series.

Proof. The proof is easy. If the stated limit exists and is finite then there are numbers M and N so that

$$\frac{a_k}{b_k} < M$$

for all $k \geq N$. This shows that $a_k \leq M b_k$ for all $k \geq N$. Consequently, applying the direct comparison test, we find that the series $\sum_{k=N}^{\infty} a_k$ converges by comparison with the convergent series $\sum_{k=N}^{\infty} M b_k$. □

(Limit Comparison Test II) Let each $a_k > 0$ and $c_k > 0$. If the terms of the series $\sum_{k=1}^{\infty} a_k$ can be compared to the terms of the series $\sum_{k=1}^{\infty} c_k$ by computing

$$\lim_{k \longrightarrow \infty} \frac{a_k}{c_k} > 0$$

and if the latter series diverges, then so does the original series.

Proof. Since the limit exists and is not zero there are numbers $\epsilon > 0$ and N so that

$$\frac{a_k}{c_k} > \epsilon$$

for all $k \geq N$. This shows that, for all $k \geq N$,

$$a_k \geq \epsilon c_k$$

□

Consequently, by the direct comparison test the series $\sum_{k=N}^{\infty} a_k$ diverges by comparison with the divergent series $\sum_{k=N}^{\infty} \epsilon c_k$.

Example 2.6.3. *We look again at the series*

$$\sum_{k=1}^{\infty} \frac{k+5}{k^3+k^2+k+1},$$

comparing it, as before, to the convergent series $\sum_{k=1}^{\infty} 1/k^2$*. This now requires computing the limit*

$$\lim_{k \longrightarrow \infty} \frac{k^2(k+5)}{k^3+k^2+k+1},$$

which elementary calculus arguments show is 1*. Since it is not infinite, the original series can now be claimed to converge by a limit comparison.*

Example 2.6.4. *Again, consider the series*

$$\sum_{k=1}^{\infty} \sqrt{\frac{k+5}{k^2+k+1}}$$

by comparing with the divergent series $\sum_{k=1}^{\infty} 1/\sqrt{k}$*. We are required to compute the limit*

$$\lim_{k \longrightarrow \infty} \sqrt{k}\sqrt{\frac{k+5}{k^2+k+1}},$$

which elementary calculus arguments show is 1*. Since it is not zero, the original series can now be claimed to diverge by a limit comparison.*

2.6.4 Ratio Comparison Test

Again we wish to compare two series $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ composed of positive terms. Rather than directly comparing the size of the terms we compare the ratios of the terms. The inspiration for this test rests on attempts to compare directly a series with a convergent geometric series. If $\sum_{k=1}^{\infty} b_k$ is a geometric series with common ratio r, then evidently

$$\frac{b_{k+1}}{b_k} = r.$$

This suggests that perhaps a comparison of ratios of successive terms would indicate how fast a series might be converging. **(Ratio Comparison Test)** If the ratios satisfy

$$\frac{a_{k+1}}{a_k} \leq \frac{b_{k+1}}{b_k}$$

for all k (or just for all k sufficiently large) and the series $\sum_{k=1}^{\infty} b_k$ with the larger ratio is convergent, then the series $\sum_{k=1}^{\infty} a_k$ is also convergent.

Proof. As usual, we assume all terms are positive in both series. If the ratios satisfy

$$\frac{a_{k+1}}{a_k} \leq \frac{b_{k+1}}{b_k}$$

for $k > N$, then they also satisfy

$$\frac{a_{k+1}}{b_{k+1}} \leq \frac{a_k}{b_k},$$

which means that the sequence $\{a_k/b_k\}$ is decreasing for $k > N$. In particular, that sequence is bounded above, say by C, and so

$$a_k \leq C b_k.$$

Thus an application of the direct comparison test shows that the series $\sum_{k=1}^{\infty} a_k$ converges. □

2.6.5 d'Alembert's Ratio Test

The ratio comparison test requires selecting a series for comparison. Often a geometric series $\sum_{k=1}^{\infty} r^k$ for some $0 < r < 1$ may be used. How do we compute a number r that will work? We would wish to use $b_k = r^k$ with a choice of r so that

$$\frac{a_{k+1}}{a_k} \leq \frac{b_{k+1}}{b_k} = \frac{r^{k+1}}{r^k} = r.$$

One useful and easy way to find whether there will be such an r is to compute the limit of the ratios.

(Ratio Test) If terms of the series $\sum_{k=1}^{\infty} a_k$ are all positive and the ratios satisfy

$$\lim_{k \longrightarrow \infty} \frac{a_{k+1}}{a_k} < 1$$

then the series $\sum_{k=1}^{\infty} a_k$ is convergent.

Proof. The proof is easy. If

$$\lim_{k \longrightarrow \infty} \frac{a_{k+1}}{a_k} < 1,$$

then there is a number $\beta < 1$ so that

$$\frac{a_{k+1}}{a_k} < \beta$$

for all sufficiently large k. Thus the series $\sum_{k=1}^{\infty}$ converges by the ratio comparison test applied to the convergent geometric series $\sum_{k=1}^{\infty} \beta^k$. □

Note. The ratio test can also be pushed to give a divergence ansour: If

$$\lim_{k \longrightarrow \infty} \frac{a_{k+1}}{a_k} > 1 \qquad (4)$$

then the series $\sum_{k=1}^{\infty} a_k$ is divergent. But it is best to downplay this test or we might think it gives an ansour as useful as the convergence test. From (4) it follows that there must be an N and β so that

$$\frac{a_{k+1}}{a_k} > \beta > 1$$

for all $k \geq N$. Then

$$a_{N+1} > \beta a_N,$$

$$a_{N+2} > \beta a_{N+1} > \beta^2 a_N,$$

and

$$a_{N+3} > \beta a_{N+2} > \beta^3 a_N.$$

We see that the terms a_k of the series are growing large at a geometric rate. Not only is the series diverging, but it is diverging in a dramatic way.

We can summarize how this test is best applied. If terms of the series $\sum_{k=1}^{\infty} a_k$ are all positive, compute

$$\lim_{k \longrightarrow \infty} \frac{a_{k+1}}{a_k} = L.$$

1. If $L < 1$, then the series $\sum_{k=1}^{\infty} a_k$ is convergent.
2. If $L > 1$, then the series $\sum_{k=1}^{\infty} a_k$ is divergent; moreover, the terms $s_k \longrightarrow \infty$.
3. If $L = 1$, then the series $\sum_{k=1}^{\infty}$ may diverge or converge, the test being inconclusive.

Example 2.6.5. *The series*

$$\sum_{k=0}^{\infty} \frac{(k!)^2}{(2k)!}$$

is particularly suited for an application of the ratio test since the ratio is easily computed and a limit taken: If we write $a_k = (k!)^2/(2k)!$, *then*

$$\frac{a_{k+1}}{a_k} = \frac{((k+1)!)^2(2k)!}{(2k+2)!(k!)^2} = \frac{(k+1)^2}{(2k+2)(2k+1)} \longrightarrow \frac{1}{4}.$$

Consequently, this is a convergent series. More than that, it is converging faster than any geometric series

$$\sum_{k=0}^{\infty} \left(\frac{1}{4} + \epsilon\right)^k$$

for any positive ϵ.

2.6.6 Cauchys Root Test

There is yet another way to achieve a comparison with a convergent geometric series. We suspect that a series $\sum_{k=1}^{\infty} a_k$ can be compared to some geometric series $\sum_{k=1}^{\infty} r^k$ but do not know how to compute the value of r that might work. The limiting values of the ratios

$$\frac{a_{k+1}}{a_k}$$

provide one way of determining what r might work but often are difficult to compute. Instead we recognize that a comparison of the form

$$a_k \leq C r^k$$

would mean that

$$\sqrt[k]{a_k} \leq \sqrt[k]{C} r.$$

For large k the term $\sqrt[k]{C}$ is close to 1, and this motivates our next test, usually attributed to Cauchy.

(Root Test) If terms of the series $\sum_{k=1}^{\infty} a_k$ are all nonnegative and if the roots satisfy

$$\lim_{k \longrightarrow \infty} \sqrt[k]{a_k} < 1,$$

then that series converges.

Proof. This is almost trivial. If

$$(a_k)^{1/k} < \beta < 1$$

for all $k \geq N$, then

$$a_k < \beta^k$$

and so $\sum_{k=1}^{\infty} a_k$ converges by direct comparison with the convergent geometric series $\sum_{k=1}^{\infty} \beta^k$. □

Again we can summarize how this test is best applied. The conclusions are nearly identical with those for the ratio test. Compute

$$\lim_{k \longrightarrow \infty} (a_k)^{1/k} = L.$$

1. If $L < 1$, then the series $\sum_{k=1}^{\infty} a_k$ is convergent.
2. If $L > 1$, then the series $\sum_{k=1}^{\infty} a_k$ is divergent; moreover, the terms $a_k \longrightarrow \infty$.
3. If $L = 1$, then the series $\sum_{k=1}^{\infty} a_k$ may diverge or converge, the test being inconclusive.

Example 2.6.6. *We found the series*

$$\sum_{k=0}^{\infty} \frac{(k!)^2}{(2k)!}$$

to be handled easily by the ratio test. It would be extremely unpleasant to attempt a direct computation using the root test. On the other hand, the series

$$\sum_{k=0}^{\infty} kx^k = x + 2x^2 + 3x^3 + 4x^4 + ...$$

for $x > 0$ *can be handled by either of these tests. we should try the ratio test while we try the root test:*

$$\lim_{k \longrightarrow \infty} \left(kx^k\right)^{1/k} = \lim_{k \longrightarrow \infty} \sqrt[k]{k}x = x$$

and so convergence can be claimed for all $0 < x < 1$ *and divergence for all* $x > 1$*. The case* $x = 1$ *is inconclusive for the root test, but the trivial test shows instantly that the series diverges for* $x = 1$.

2.6.7 Cauchy's Condensation Test

Occasionally a method that is used to study a specific series can be generalized into a useful test. Recall that in studying the sequence of partial sums of the harmonic series it was convenient to watch only at the steps $1, 2, 4, 8, ...$ and make a rough loour estimate. The reason this worked was simply that the terms in the harmonic series decrease and so estimates of $s_1, s_2, s_4, s_8, ...$ oure easy to obtain using just that fact. This turns quickly into a general test.

(Cauchy's Condensation Test) If the terms of a series $\sum_{k=1}^{\infty} a_k$ are nonnegative and decrease monotonically to zero, then that series con- verges if and only if the related series

$$\sum_{j=1}^{\infty} 2^j a_{2^j}$$

converges.

Proof. Since all terms are nonnegative, we need only compare the size of the partial sums of the two series. Computing first the sum of $2^{p+1} - 1$ terms of the original series, we have

$$a_1 + (a_2 + a_3) + \cdots + (a_{2^p} + a_{2^p+1} + \cdots + a_{2^{p+1}1})$$

$$\leq a_1 + 2a_2 + \cdots + 2^p a_{2^p}.$$

And, with the inequality sign in the opposite direction, we compute the sum of 2^p terms of the original series to obtain

$$a_1 + a_2 + (a_3 + a_4) + \cdots + (a_{2^{p-1}+1} + a_{2^{p-1}+2} + \cdots + a_{2^p})$$

$$\geq \frac{1}{2}(a_1 + 2a_2 + \cdots + 2^p a_{2^p}).$$

If either series has a bounded sequence of partial sums so too then does the other series. Thus both converge or else both diverge. ☐

Example 2.6.7. *Let us use this test to study the p-harmonic series:*

$$\sum_{k=1}^{\infty} \frac{1}{k^p}$$

for $p > 0$. The terms decrease to zero and so the convergence of this series is equivalent to the convergence of the series

$$\sum_{j=1}^{\infty} 2^j \left(\frac{1}{2^j}\right)^p$$

and this series is a geometric series

$$\sum_{j=1}^{\infty} (2^{1-p})^j.$$

This converges precisely when $2^{1p} < 1$ or $p > 1$ and diverges when $2^{1p} \geq 1$ or $p \leq 1$. Thus we know exactly the convergence behavior of the p-harmonic series for all values of p. (For $p \leq 0$ we have divergence just by the trivial test.)

It is worth deriving a simple test from the Cauchy condensation test as a corollary. This is an improvement on the trivial test. The trivial test requires that $\lim_{k\longrightarrow\infty} ka_k = 0$ for a convergent series $\sum_{k=1}^{\infty} a_k$. This next test, which is due to Abel, shows that slightly more can be said if the terms form a monotonic sequence. The sequence $\{a_k\}$ must go to zero faster than $\{1/k\}$.

Corollary 2.6.8. *If the terms of a convergent series $\sum_{k=1}^{\infty} a_k$ decrease monotonically, then*

$$\lim_{k\longrightarrow\infty} ka_k = 0.$$

Proof. By the Cauchy condensation test we know that

$$\lim_{j\longrightarrow\infty} 2^j a_{2^j} = 0.$$

If $2^j \le k \le 2^{j+1}$, then $a_k \le a_{2^j}$ and so

$$ka_k \le 2(2^j a_{2^j}),$$

which is small for large j. Thus $ka_k \longrightarrow 0$ as required. □

2.6.8 Integral Test

To determine the convergence of a series $\sum_{k=1}^{\infty} a_k$ of nonnegative terms it is often necessary to make some kind of estimate on the size of the sequence of partial sums. Most of our tests have done this automatically, saving us the labor of computing such estimates. Sometimes those estimates can be obtained by calculus methods. The integral test allows us to estimate the partial sums $\sum_{k=1}^{n} f(k)$ by computing instead $\int_1^n f(x)$ in certain circumstances. This is more than a convenience; it also shows a close relation between series and infinite integrals, which is of much importance in analysis.

(Integral Test) Let f be a nonnegative decreasing function on $[1, \infty)$ such that the integral $\int_1^X f(x)dx$ can be computed for all $X > 1$. If

$$\lim_{X\longrightarrow\infty} \int_1^X f(x)dx < \infty$$

exists, then the series $\sum_{k=1}^{\infty} f(k)$ converges. If

$$\lim_{X\longrightarrow\infty} \int_1^X f(x)dx = \infty,$$

then the series $\sum_{k=1}^{\infty}$ diverges.

Proof. Since the function f is decreasing we must have

$$\int_k^{k+1} f(x)dx \le f(k) \le \int_{k-1}^{k} f(x)dx.$$

Applying these inequalities for $k = 2, 3, 4, \ldots$ we obtain

$$\int_1^{n+1} f(x)dx \le \sum_{k=1}^{n} f(k) \le f(1) + \int_1^n f(x)dx. \qquad (5)$$

The series converges if and only if the partial sums are bounded. But we see from the inequalities (5) that if the limit of the integral is finite, then these partial sums are bounded. If the limit of the integral is infinite, then these partial sums are unbounded. □

Note. The convergence of the integral yields the convergence of the series. There is no claim that the sum of the series $\sum_{k=1}^{\infty} f(k)$ and the value of the infinite integral $\int_1^{\infty} f(x)dx$ are the same.

Example 2.6.9. *According to this test the harmonic series* $\sum_{k=1}^{\infty} \frac{1}{k}$ *can be studied by computing*

$$\lim_{X \longrightarrow \infty} \int_1^X \frac{dx}{x} = \lim_{X \longrightarrow \infty} \log X = \infty.$$

For the same reasons the p-harmonic series

$$\sum_{k=1}^{\infty} \frac{1}{k^p}$$

for $p > 1$ *can be studied by computing*

$$\lim_{X \longrightarrow \infty} \int_1^X \frac{dx}{x^p} = \lim_{X \longrightarrow \infty} \frac{1}{p-1}\left(1 - \frac{1}{X^{p-1}}\right) = \frac{1}{p-1}.$$

In both cases we obtain the same conclusion as before. The harmonic series diverges and, for $p > 1$*, the p-harmonic series converges.*

2.6.9 Kummer's Tests

The ratio test requires merely taking the limit of the ratios

$$\frac{a_{k+1}}{a_k}$$

but often fails. We know that if this tends to 1, then the ratio test says nothing about the convergence or divergence of the series $\sum_{k=1}^{\infty} a_k$.

Kummer's tests provide a collection of ratio tests that can be designed by taking different choices of sequence $\{D_k\}$. The choices $D_k = 1, D_k = k$ and $D_k = k \ln k$ are used in the following tests. Ernst Eduard Kummer (18101893) is probably most famous for his contributions to the study of Fermat's last theorem; his tests arose in his study of hypergeometric series.

(Kummer's Tests) The series $\sum_{k=1}^{\infty} a_k$ can be tested by the following criteria. Let $\{D_k\}$ denote any sequence of positive numbers and compute

$$L = \lim_{k \longrightarrow \infty} \inf \left[D_k \frac{a_k}{a_{k+1}} - D_{k+1} \right].$$

If $L > 0$ the series $\sum_{k=1}^{\infty} a_k$ converges. On the other hand, if

$$\left[D_k \frac{a_k}{a_{k+1}} - D_{k+1} \right] \leq 0$$

for all sufficiently large k and if the series

$$\sum_{k=1}^{\infty} \frac{1}{D_k}$$

diverges, then the series $\sum_{k=1}^{\infty} a_k$ diverges.

Proof. If $L > 0$, then we can choose a positive number $\alpha < L$. By the definition of a lim inf this means there must exist an integer N so that for all $k \geq N$,

$$\alpha < \left[D_k \frac{a_k}{a_{k+1}} - D_{k+1}\right].$$

Rewriting this, we find that

$$\alpha a_{k+1} < D_k a_k - D_{k+1} a_{k+1}.$$

We can write this inequality for $k = N, N+1, N+2, ...N+p$ to obtain

$$\alpha a_{k+1} < D_N a_N - D_{N+1} a_{N+1}$$

$$\alpha a_{k+2} < D_{N+1} a_{N+1} - D_{N+2} a_{N+2}$$

and so on. Adding these (note the telescoping sums), we find that

$$\alpha(a_{N+1} + a_{N+2} + \cdots + a_{N+p+1})$$

$$< D_{N+1} a_{N+1} D_{N+p+1} a_{N+p+1} < D_{N+1} a_{N+1}.$$

(The final inequality just uses the fact that all the terms here are positive.)

From this inequality we can determine that the partial sums of the series $\sum_{k=1}^{\infty} a_k$ are bounded. By our usual criterion, this proves that this series converges.

The second part of the theorem requires us to establish divergence. Suppose now that

$$D_k \frac{a_k}{a_{k+1}} - D_{k+1} \leq 0$$

for all $k \geq N$. Then

$$D_k a_k \leq D_{k+1} a_{k+1}.$$

Thus the sequence $\{D_k a_k\}$ is increasing after $k = N$. In particular,

$$D_k a_k \geq C$$

for some C and all $k \geq N$ and so

$$a_k \geq \frac{C}{D_k}.$$

It follows by a direct comparison with the divergent series $\sum C/D_k$ that our series also diverges. □

Note. In practice, for the divergence part of the test, it may be easier to compute

$$L = \lim_{k \longrightarrow \infty} \sup \left[D_k \frac{a_k}{a_{k+1}} - D_{k+1}\right].$$

If $L < 0$, then we would know that

$$\left[D_k \frac{a_k}{a_{k+1}} - D_{k+1}\right] \leq 0$$

for all sufficiently large k and so, if the series $\sum_{k=1}^{\infty} \frac{1}{D_k}$ diverges, then the series $\sum_{k=1}^{\infty} a_k$ diverges.

Example 2.6.10. *What is Kummer's test if the sequence used is the simplest possible $D_k = 1$ for all k? In this case it is simply the ratio test. For example, suppose that*

$$\lim_{k \longrightarrow \infty} \frac{a_{k+1}}{a_k} = r.$$

Then, replacing $D_k = 1$, we have

$$\lim_{k \longrightarrow \infty} \left[D_k \frac{a_k}{a_{k+1}} - D_{k+1} \right] = \lim_{k \longrightarrow \infty} \left[\frac{a_k}{a_{k+1}} - 1 \right] = \frac{1}{r} - 1.$$

Thus, by Kummer's test, if $1/(r-1) < 0$ we have divergence while if $1/(r-1) > 0$ we have convergence. These are just the cases $r > 1$ and $r < 1$ of the ratio test.

2.6.10 Raabe's Ratio Test

A simple variant on the ratio test is known as Raabe's test. Suppose that

$$\lim_{k \longrightarrow \infty} \frac{a_k}{a_{k+1}} = 1$$

so that the ratio test is inconclusive. Then instead compute

$$\lim_{k \longrightarrow \infty} k \left(\frac{a_k}{a_{k+1}} - 1 \right).$$

The series $\sum_{k=1}^{\infty} a_k$ converges or diverges depending on whether this limit is greater than or less than 1.

(Raabe's Test) The series $\sum_{k=1}^{\infty} a_k$ can be tested by the following criterion. Compute

$$L = \lim_{k \longrightarrow \infty} k \left(\frac{a_k}{a_{k+1}} - 1 \right).$$

Then

1. If $L > 1$, the series $\sum_{k=1}^{\infty} a_k$ converges.
2. If $L < 1$, the series $\sum_{k=1}^{\infty} a_k$ diverges.
3. If $L = 1$, the test is inconclusive.

Proof. This is Kummer's test using the sequence $D_k = k$. □

Example 2.6.11. *Consider the series*

$$\sum_{k=0}^{\infty} \frac{k^k}{e^k k!}.$$

An attempt to apply the ratio test to this series will fail since the ratio will tend to 1, the inconclusive case. But if instead we consider the limit

$$\lim_{k \longrightarrow \infty} k \left(\left(\frac{k^k}{e^k k!} \right) \left(\frac{e^{k+1}(k+1)!}{(k+1)^{k+1}} \right) - 1 \right)$$

as called for in Raabe's test, we can use calculus methods (L'Hopital's rule) to obtain a limit of $\frac{1}{2}$. Consequently, this series diverges.

2.6.11 Gauss's Ratio Test

Raabe's test can be replaced by a closely related test due to Gauss. We might have discovered while using Raabe's test that

$$\lim_{k\longrightarrow\infty} k\left(\frac{a_k}{a_{k+1}}-1\right)=L.$$

This suggests that in any actual computation we will have discovered, perhaps by division, that

$$\frac{a_k}{a_{k+1}}=1+\frac{L}{k}+\text{terms involving}\quad\frac{1}{k^2}\quad etc.$$

The case $L>1$ corresponds to convergence and the case $L<1$ to divergence, both by Raabe's test. What if $L=1$, which is considered inconclusive in Raabes test?

Gauss's test offers a different way to look at Raabe's test and also has an added advantage that it handles this case that was left as inconclusive in Raabe's test.

(Gausss Test) The series $\sum_{k=1}^{\infty} a_k$ can be tested by the following criterion. Suppose that

$$\frac{a_k}{a_{k+1}}=1+\frac{L}{k}+\frac{\phi(k)}{k^2}$$

where $\phi(k)(k=1,2,3,...)$ forms a bounded sequence. Then

1. If $L>1$ the series $\sum_{k=1}^{\infty} a_k$ converges.
2. If $L\leq 1$ the series $\sum_{k=1}^{\infty} a_k$ diverges.

Proof. As we noted, for $L>1$ and $L<1$ this is precisely Raabe's test. Only the case $L=1$ is new! Let us assume that

$$\frac{a_k}{a_{k+1}}=1+\frac{1}{k}+\frac{x_k}{k^2}$$

where $\{a_k\}$ is a bounded sequence.

To prove this case (that the series diverges) we shall use Kummer's test with the sequence $D_k=k\log k$. We consider the expression

$$\left[D_k\frac{a_k}{a_{k+1}}-D_{k+1}\right],$$

which now assumes the form

$$k\log k\frac{a_k}{a_{k+1}}-(k+1)\log(k+1)$$

$$=k\log k\left(1+\frac{1}{k}+\frac{x_k}{k^2}\right)-(k+1)\log(k+1).$$

We need to compute the limit of this expression as $k\longrightarrow\infty$. It takes only a few manipulations (which we should try) to see that the limit is -1. For this use the facts that

$$(\log k)/k\longrightarrow 0$$

and

$$(k+1)\log(1+1/k)\longrightarrow 1$$

as $k \longrightarrow \infty$.

We are now in a position to claim, by Kummer's test, that our series $\sum_{k=1}^{\infty} a_k$ diverges. To apply this part of the test requires us to check that the series

$$\sum_{k=2}^{\infty} \frac{1}{k \log k}$$

diverges. Several tests would work for this. Perhaps Cauchys condensation test is the easiest to apply, although the integral test can be used too. □

Note. In Gausss test we may be puzzling over how to obtain the expression

$$\frac{a_k}{a_{k+1}} = 1 + \frac{L}{k} + \frac{\phi(k)}{k^2}.$$

In practice often the fraction a_k/a_{k+1} is a ratio of polynomials and so usual algebraic procedures will supply this. In theory, though, there is no problem. For any L we could simply write

$$\phi(k) = k^2 \left(\frac{a_k}{a_{k+1}} - 1 + \frac{L}{k} \right).$$

Thus the real trick is whether it can be done in such a way that the $\phi(k)$ do not grow too large.

Also, in some computations we might prefer to leave the ratio as ak+1/ak the way it was for the ratio test. In that case Gausss test would assume the form

$$\frac{a_{k+1}}{a_k} = 1 - \frac{L}{k} + \frac{\phi(k)}{k^2}.$$

(Note the minus sign.) The conclusions are exactly the same.

Example 2.6.12. *The series*

$$1 + mx + \frac{m(m-1)}{2!}x^2 +$$
$$\frac{m(m-1)(m-2)}{3!}x^3 + \frac{m(m-1)...(m-k+1)}{k!}x^k + ...$$

is called the binomial series. When m is a positive integer all terms for $k > m$ are zero and the series reduces to the binomial formula for $(1+x)^m$. Here now m is any real number and the hope remains that the formula might still be valid, but using a series rather than a finite sum. This series plays an important role in many applications. Let us check for absolute convergence at $x = 1$. We can assume that $m \neq 0$ since that case is trivial.

If we call the absolute value of the $k+1$st term a_k so

$$a_{k+1} = \left| \frac{m(m-1)...(m-k+1)}{k!} \right|,$$

then a simple calculation shows that for large values of k

$$\frac{a_{k+1}}{a_k} = 1 - \frac{m+1}{k}.$$

Here we are using the version a_{k+1}/a_k rather than the reciprocal; see the preceding note.

There are no higher-order terms to worry about in Gausss test here and so the series $\sum a_k$ converges if $m+1 > 1$ and diverges if $m+1 < 1$. Thus the binomial series converges absolutely for $x = 1$ if $m > 0$. For $m = 0$ the series certainly converges since all terms except for the first one are identically zero. For $m < 0$ we know so far only that it does not converge absolutely. A closer analysis, for those who might care to try, will show that the series is non absolutely convergent for $1 < m < 0$ and divergent for $m \leq 1$.

2.6.12 Alternating Series Test

We pass now to a number of tests that are needed for studying series of terms that may change signs. The simplest first step in studying a series $\sum_{i=1}^{\infty} a_i$, where the a_i are both negative and positive, is to apply one from our battery of tests to the series $\sum_{i=1}^{\infty} |a_i|$. If any test shows that this converges, then we know that our original series converges absolutely. This is even better than knowing it converges.

But what shall we do if the series is not absolutely convergent or if such attempts fail? One method applies to special series of positive and negative terms. Recall how we handled the series

$$\sum_{k=1}^{\infty}(-1)^{k-1}\frac{1}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4}...$$

(called the alternating harmonic series). We considered separately the partial sums $s_2, s_4, s_6, ...$ and $s_1, s_3, s_5, ...$ The special pattern of $+$ and $-$ signs alternating one after the other allowed us to see that each subsequence $\{s_{2n}\}$ and $\{s_{2n-1}\}$ was monotonic. All the features of this argument can be put into a test that applies to a wide class of series, similar to the alternating harmonic series.

(Alternating Series Test) The series

$$\sum_{k=1}^{\infty}(-1)^{k-1}a_k,$$

whose terms alternate in sign, converges if the sequence $\{a_k\}$ decreases monotonically to zero. Moreover, the value of the sum of such a series lies between the values of the partial sums at any two consecutive stages.

Proof. The proof is just exactly the same as for the alternating harmonic series. Since the a_k are nonnegative and decrease, we compute that

$$a_1 a_2 = s_2 \leq s_4 \leq s_6 \leq \cdot \leq s_5 \leq s_3 \leq s_1 = a_1.$$

These subsequences then form bounded monotonic sequences and so

$$\lim_{n\longrightarrow\infty} s_{2n} \quad \text{and} \quad \lim_{n\longrightarrow\infty} s_{2n-1}$$

exist. Finally, since

$$s_{2n} - s_{2n-1} = -a_{2n} \longrightarrow 0$$

we can conclude that $\lim_{n\longrightarrow\infty} s_n = L$ exists. From the proof it is clear that the value L lies in each of the intervals $[s_2, s_1], [s_2, s_3], [s_4, s_3], [s_4, s_5], ...$ and so, as stated, the sum of the series lies between the values of the partial sums at any two consecutive stages. □

2.6.13 Dirichlets Test

Our next test derives from the summation by parts formula

$$\sum_{k=1}^{n} a_k b_k = s_1(b_1 - b_2) + s_2(b_2 - b_3) + \cdots + s_{n-1}(b_{n-1} - b_n) + s_n b_n$$

We can see that if there is some special information available about the sequences $\{s_n\}$ and $\{b_n\}$ here, then the convergence of the series $\sum_{k=1}^{n} a_k b_k$ can be proved. The test gives one possibility for this. The next section gives a different variant.

The test is named after Lejeune Dirichlet (18051859) who is most famous for his work on Fourier series, in which this test plays an important role.

(Dirichlet Test) If $\{b_n\}$ is a sequence decreasing to zero and the partial sums of the series $\sum_{k=1}^{\infty} a_k$ are bounded, then the series $\sum_{k=1}^{\infty} a_k b_k$ converges.

Proof. Write $s_n = \sum_{k=1}^{n} a_k$. By our assumptions on the series $\sum_{k=1}^{\infty} a_k$ there is a positive number M so that $|s_n| \leq M$ for all n. Let $\epsilon > 0$ and choose N so large that $b_n < \epsilon/(2M)$ if $n \geq N$.

The summation by parts formula shows that for $m > n \geq N$

$$\left|\sum_{k=n}^{m} a_k b_k\right| = |a_n b_n + a_{n+1} b_{n+1} + \cdots + a_m b_m|$$

$$= |-s_{n-1} b_n + s_n(b_n - b_{n+1}) + \cdots + s_{m-1}(b_{m-1} - b_m) + s_m b_m|$$

$$\leq |-s_{n-1} b_n| + |s_n(b_n - b_{n+1})| + \cdots + |s_{m-1}(b_{m-1} - b_m)| + |s_m b_m|$$

$$\leq M(b_n + [b_n - b_m] + b_m) \leq 2M b_n < \epsilon.$$

Notice that we have needed to use the fact that

$$b_{k1} b_k \geq 0$$

for each k. This is precisely the Cauchy criterion for the series $\sum_{k=1}^{\infty} a_k b_k$ and so we have proved convergence. □

Example 2.6.13. *The series*

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} \ldots$$

converges by the alternating series test. What other pattern of + and − signs could we insert and still have convergence? Let $a_k = \pm 1$. If the partial sums

$$\sum_{k=1}^{n} a_k$$

remain bounded, then, by Dirichlets test, the series

$$\sum_{k=1}^{n} \frac{a_k}{k}$$

must converge. Thus, for example, the pattern

$$+ - + + - - + - + + - - + - + + - - \ldots$$

would produce a convergent series (that is not alternating).

2.6.14 Abel's Test

The next test is another variant on the same theme as the Dirichlet test. There the series $\sum_{k=1}^{\infty} a_k b_k$ was proved to be convergent by assuming a fairly weak fact for the series $\sum_{k=1}^{\infty} a_k$ (i.e., bounded partial sums) and a strong fact for $\{b_k\}$ (i.e., monotone convergence to 0). Here we strengthen the first and weaken the second.

(Abel Test) If $\{b_n\}$ is a convergent monotone sequence and the series $\sum_{k=1}^{\infty} a_k$ is convergent, then the series $\sum_{k=1}^{\infty} a_k b_k$ converges.

Proof. Suppose first that b_k is decreasing to a limit B. Then $b_k - B$ decreases to zero. We can apply Dirichlets test to the series

$$\sum_{k=1}^{\infty} a_k(b_k - B)$$

to obtain convergence, since if $\sum_{k=1}^{\infty} a_k$ is convergent, then it has a bounded sequence of partial sums.

But this allows us to express our series as the sum of two convergent series:

$$\sum_{k=1}^{\infty} a_k b_k = \sum_{k=1}^{\infty} a_k(b_k - B) + B\sum_{k=1}^{\infty} a_k.$$

If the sequence b_k is instead increasing to some limit then we can apply the first case proved to the series $-\sum_{k=1}^{\infty} a_k(-b_k)$. □

2.7 Rearrangements

[2]Any finite sum may be rearranged and summed in any order. This is because addition is commutative. We might expect the same to occur for series. We add up a series $\sum_{k=1}^{\infty} a_k$ by starting at the first term and adding in the order presented to us. If the terms are rearranged into a dierent order do we get the same result?

Example 2.7.1. *The most famous example of a series that cannot be freely rearranged without changing the sum is the alternating harmonic series. We know that the series*

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4}...$$

is convergent (actually non absolutely convergent) with a sum somewhere between $1/2$ *and* 1. *If we rearrange this so that every positive term is followed by two negative terms, thus,*

$$1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \frac{1}{10} - \frac{1}{12}...$$

we shall arrive at a different sum. Grouping these and adding, we obtain

$$\left(1 - \frac{1}{2}\right) - \frac{1}{4} + \left(\frac{1}{3} - \frac{1}{6}\right) - \frac{1}{8} + \left(\frac{1}{5} - \frac{1}{10}\right) - \frac{1}{12}...$$

$$= \frac{1}{2}\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4}...\right)$$

whose sum is half the original series. Rearranging the series has changed the sum!

For the theory of unordered sums there is no such problem. If an unordered sum $\sum_{j\in J} a_j$ converges to a number c, then so too does any rearrangement. If $\sigma : I \longrightarrow I$ is one-to-one and onto, then

$$\sum_{i\in I} a_j = \sum_{i\in I} a_{\sigma(i)}.$$

We had hoped for the same situation for series. If $\sigma : \mathbb{N} \longrightarrow \mathbb{N}$ is one-to-one and onto, then

$$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} a_{\sigma(k)}$$

may or may not hold. We call $\sum_{k=1}^{\infty} a_{\sigma(k)}$ a rearrangement of the series $\sum_{k=1}^{\infty} a_k$.

We propose now to characterize those series that allow unlimited rearrangements, and those that are more fragile (as is the alternating harmonic series) and cannot permit rearrangement.

2.7.1 Unconditional Convergence

A series is said to be unconditionally convergent if all rearrangements of that series converge and have the same sum. Those series that do not allow this but do converge are called conditionally convergent. Here the conditional means that the series converges in the arrangement given, but may diverge in another arrangement or may converge to a different sum in another arrangement. We shall see that conditionally convergent series are extremely fragile; there are rearrangements that exhibit any behavior desired. There are rearrangements that diverge and there are rearrangements that converge to any desired number.

Our first theorem asserts that any absolutely convergent series may be freely rearranged. All absolutely convergent series are unconditionally convergent. In fact, the two terms are equivalent

unconditionally convergent $\Longleftrightarrow$ *absolutely convergent*

although we must wait until the next section to prove that.

Theorem 2.7.2. ***(Dirichlet)*** *Every absolutely convergent series is unconditionally convergent.*

Proof. Let us prove this first for series $\sum_{k=1}^{\infty} a_k$ a whose terms are all non negative. For such series convergence and absolute convergence mean the same thing.

Let $\sum_{k=1}^{\infty} a_{\sigma(k)}$ be any rearrangement. Then for any M

$$\sum_{k=1}^{M} a_{\sigma(k)} \leq \sum_{k=1}^{N} a_k \leq \sum_{k=1}^{\infty} a_k$$

by choosing an N large enough so that $\{1, 2, 3, ..., N\}$ includes all the integers $\{\sigma(1), \sigma(2), \sigma(3), ..., \sigma(M)\}$. By the bounded partial sums criterion this shows that $\sum_{k=1}^{\infty} a_{\sigma(k)}$ is convergent and to a sum smaller than $\sum_{k=1}^{\infty} a_k$. But this same argument would show that $\sum_{k=1}^{\infty} a_k$ is convergent and to a sum smaller than $\sum_{k=1}^{\infty} a_{\sigma(k)}$ and consequently all rearrangements converge to the same sum.

We now allow the series $\sum_{k=1}^{\infty} a_k$ to have positive and negative values.
Write

$$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} [a_k]^+ - \sum_{k=1}^{\infty} [a_k]^-$$

where we are using the notation

$$[X]^+ = \max\{X, 0\} \quad \text{and} \quad [X]^- = \max\{X, 0\}$$

and remembering that

$$X = [X]^+[X]^- \quad \text{and} \quad |X| = [X]^+ + [X]^-.$$

Any rearrangement of the series on the left-hand side of this identity just results in a rearrangement in the two series of non negative terms on the right. We have just seen that this does nothing to alter the convergence or the sum. Consequently, any rearrangement of our series will have the same sum as required to prove the assertion of the theorem. □

2.7.2 Conditional Convergence

A convergent series is said to be conditionally convergent if it is not unconditionally convergent. Thus such a series converges in the arrangement given, but either there is some rearrangement that diverges or else there is some rearrangement that has a different sum. In fact, both situations always occur.

We shall show that any non absolutely convergent series has this property. Our previous rearrangement took advantage of the special nature of the series; here our proof must be completely general and so the method is different.

The following theorem completes Theorem 3.48 and provides the connections:

$$\textit{conditionally convergent} \iff \textit{nonabsolutely convergent}$$

and

$$\textit{unconditionally convergent} \iff \textit{absolutely convergent}$$

Note. we may wonder why we have needed this extra terminology if these concepts are identical. One reason is to emphasize that this is part of the theory. Conditional convergence and nonabsolutely convergence may be equivalent, but they have different underlying meanings. Also, this terminology is used for series of other objects than real numbers and for series of this more general type the terms are not equivalent.

Theorem 2.7.3. ***(Riemann)*** *Every nonabsolutely convergent series is conditionally convergent. In fact, every nonabsolutely convergent series has a divergent rearrangement and can also be rearranged to sum to any preassigned value.*

Proof. Let $\sum_{k=1}^{\infty}$ be an arbitrary nonabsolutely convergent series. To prove the first statement it is enough if we observe that both series

$$\sum_{k=1}^{\infty}[a_k]^+ \quad \text{and} \quad \sum_{k=1}^{\infty}[a_k]^-$$

must diverge in order for $\sum_{k=1}^{\infty} a_k$ to be nonabsolutely convergent. We need to observe as well that $a_k \longrightarrow 0$ since the series is assumed to be convergent.

Write $p_1, p_2, p_3,$ for the sequence of positive numbers in the sequence $\{a_k\}$ (skipping any zero or negative ones) and write $q_1, q_2, q_3, ...$ for the sequence of terms that we have skipped. We construct a new series

$$p_1 + p_2 + \ + p_{n_1} + q_1 + p_{n_1+1} + p_{n_1+2} + \ + p_{n_2} + q_2 + p_{n_2+1} \cdots$$

where we have chosen $0 = n_0 < n_1 < n_2 < n_3 ...$ so that

$$p_{n_{k+1}} + p_{n_1+2} + ... + p_{n_{k+1}} > 2^k$$

for each $k = 0, 1, 2, \ldots.$ Since $\sum_{k=1}^{\infty} p_k$ diverges, this is possible. The new series so constructed contains all the terms of our original series and so is a rearrangement. Since the terms $q_k \longrightarrow 0$, they will not interfere with the goal of producing ever larger partial sums for the new series and so, evidently, this new series diverges to ∞.

The second requirement of the theorem is to produce a convergent rearrangement, convergent to a given number α. We proceed in much the same way but with rather more caution. We leave this to the exercises. □

2.7.3 Comparison of $\sum_{i=1}^{\infty} a_i$ and $\sum_{i\in\mathbb{N}} a_i$

The unordered sum of a sequence of real numbers, written as,

$$\sum_{i\in\mathbb{N}} a_i,$$

has an apparent connection with the ordered sum

$$\sum_{i=1}^{\infty} a_i.$$

We should expect the two to be the same when both converge, but is it possible that one converges and not the other?

The ansour is that the convergence of $\sum_{i\in\mathbb{N}} a_i$ is equivalent to the absolute convergence of $\sum_{i=1}^{\infty} a_i$.

Theorem 2.7.4. *A necessary and sufficient condition for $\sum_{i\in\mathbb{N}} a_i$ to converge is that the series $\sum_{i=1}^{\infty} a_i$ is absolutely convergent and in this case*

$$\sum_{i\in\mathbb{N}} a_i = \sum_{i=1}^{\infty} a_i.$$

Proof. We shall use a device we have seen before a few times: For any real number X write

$$[X]^+ = \max\{X, 0\} \quad \text{and} \quad [X]^- = \max\{-X, 0\}$$

and note that

$$X = [X]^+[X]^- \quad \text{and} \quad |X| = [X]^+ + [X]^-.$$

The absolute convergence of the series and the convergence of the sum in the statement in the theorem now reduce to considering the equality of the right-hand sides of

$$\sum_{i\in\mathbb{N}} a_i = \sum_{i\in\mathbb{N}} [a_i]^+ - \sum_{i\in\mathbb{N}} [a_i]^-$$

and

$$\sum_{i=1}^{\infty} a_i = \sum_{i=1}^{\infty} [a_i]^+ - \sum_{i=1}^{\infty} [a_i]^-.$$

This reduces our problem to considering just nonnegative series (sums). Thus we may assume that each $a_i \geq 0$. For any finite set $I \subset \mathbb{N}$ it is clear that

$$\sum_{i\in I} a_i \leq \sum_{i=1}^{\infty} a_i.$$

It follows that if $\sum_{i=1}^{\infty} a_i$ converges, then so too does $\sum_{i\in\mathbb{N}} a_i$ and

$$\sum_{i\in\mathbb{N}} a_i \leq \sum_{i=1}^{\infty} a_i. \qquad (6)$$

Similarly, if N is finite,

$$\sum_{i=1}^{N} a_i \leq \sum_{i\in\mathbb{N}} a_i.$$

It follows that if $\sum_{i\in\mathbb{N}} a_i$ converges, then, by the boundedness criterion, so too does $\sum_{i=1}^{\infty} a_i$ and

$$\sum_{i=1}^{\infty} a_i \leq \sum_{i\in\mathbb{N}} a_i. \qquad (7)$$

Together these two assertions and the equations (6) and (7) prove the theorem for the case of non negative series (sums). □

2.8 Products of Series

[1]The rule for the sum of two convergent series

$$\sum_{k=0}^{\infty}(a_k + b_k) = \sum_{k=0}^{\infty} a_k + \sum_{k=0}^{\infty} b_k$$

is entirely elementary to prove and comes directly from the rule for limits of sums of sequences. If A_n and B_n represent the sum of $n+1$ terms of the two series, then

$$\lim_{n\longrightarrow\infty} \sum_{k=0}^{\infty}(a_k + b_k) = \lim_{n\longrightarrow\infty}(A_n + B_n) = \lim_{n\longrightarrow\infty} A_n + \lim_{n\longrightarrow\infty} B_n$$

$$= \sum_{k=0}^{\infty} a_k + \sum_{k=0}^{\infty} b_k.$$

At first glance we might expect to have a similar rule for products of series, since

$$\lim_{n\longrightarrow\infty}(A_n \times B_n) = \lim_{n\longrightarrow\infty} A_n \times \lim_{n\longrightarrow\infty} B_n$$

$$= \sum_{k=0}^{\infty} a_k \times \sum_{k=0}^{\infty} b_k.$$

But what is A_nB_n? If we write out this product we obtain

$$A_nB_n = (a_0 + a_1 + a_2 + \cdots + a_n)(b_0 + b_1 + b_2 + \cdots + b_n)$$

$$= \sum_{i=0}^{n}\sum_{j=1}^{n} a_ib_j.$$

From this all we can show is the curious observation that

$$\lim_{n\longrightarrow\infty} \sum_{i=0}^{n}\sum_{j=1}^{n} a_ib_j = \sum_{k=0}^{\infty} a_k \times \sum_{k=0}^{\infty} b_k.$$

What we would rather see here is a result similar to the rule for sums:

"series + series = series."

Can this result be interpreted as

"series $\times$ series = series?"

We need a systematic way of adding up the terms $a_i b_j$ in the double sum so as to form a series. The terms are displayed in a rectangular array in Figure 3.3.

If we replace the series here by a poour series, this systematic way will become much clearer. How should we add up

$$(a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n)(b_0 + b_1 x + b_2 x^2 + \cdots + b_n x^n)$$

$\times$	a_0	a_1	a_2	a_3	a_4	a_5	$\ldots$
b_0	a_0b_0	a_1b_0	a_2b_0	a_3b_0	a_4b_0	a_5b_0	$\ldots$
b_1	a_0b_1	a_1b_1	a_2b_1	a_3b_1	a_4b_1	a_5b_1	$\ldots$
b_2	a_0b_2	a_1b_2	a_2b_2	a_3b_2	a_4b_2	a_5b_2	$\ldots$
b_3	a_0b_3	a_1b_3	a_2b_3	a_3b_3	a_4b_3	a_5b_3	$\ldots$
b_4	a_0b_4	a_1b_4	a_2b_4	a_3b_4	a_4b_4	a_5b_4	$\ldots$
b_5	a_0b_5	a_1b_5	a_2b_5	a_3b_5	a_4b_5	a_5b_5	$\ldots$
$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	

Figure 3.3. The product of the two series $\sum_0^\infty a_k$ and $\sum_0^\infty b_k$.

(which with $x = 1$ is the same question we just asked)? The now obvious ansour is

$$a_0b_0 + (a_0b_1 + a_1b_0)x + (a_0b_2 + a_1b_1 + a_2b_0)x^2$$
$$+(a_0b_3 + a_1b_2 + a_2b_1 + a_3b_0)x^3 + \ldots.$$

Notice that this method of grouping the terms corresponds to summing along diagonals of the rectangle in Figure 3.3.

This is the source of the following definition.

Definition 2.8.1. *The series*

$$\sum_{k=0}^{\infty} c_k$$

is called the formal product of the two series

$$\sum_{k=0}^{\infty} a_k \quad \textit{and} \quad \sum_{k=0}^{\infty} b_k$$

provided that

$$c_k = \sum_{i=0}^{k} a_i b_{k-i}.$$

Our main goal now is to determine if this "formal" product is in any way a genuine product; that is, if

$$\sum_{k=0}^{\infty} c_k = \sum_{k=0}^{\infty} a_k \times \sum_{k=0}^{\infty} b_k.$$

The reason we expect this might be the case is that the series $\sum_{k=0}^\infty c_k$ contains all the terms in the expansion of

$$(a_0 + a_1 + a_2 + a_3 + \ldots)(b_0 + b_1 + b_2 + b_3 + \ldots).$$

A good reason for caution, however, is that the series $\sum_{k=0}^\infty c_k$ contains these terms only in a particular arrangement and we know that series can be sensitive to rearrangement.

2.8.1 Products of Absolutely Convergent Series

It is a general rule in the study of series that absolutely convergent series permit the best theorems. We can rearrange such series freely as we have seen already in Section 2.7.1. Now we show that we can form products of such series. We shall have to be much more cautious about forming products of nonabsolutely convergent series.

Theorem 2.8.2. ***(Cauchy)*** *Suppose that $\sum_{k=0}^{\infty} c_k$ is the formal product of two absolutely convergent series*

$$\sum_{k=0}^{\infty} a_k \quad \textit{and} \quad \sum_{k=0}^{\infty} b_k.$$

Then $\sum_{k=0}^{\infty} c_k$ converges absolutely too and

$$\sum_{k=0}^{\infty} c_k = \sum_{k=0}^{\infty} a_k \times \sum_{k=0}^{\infty} b_k.$$

Proof. We write

$$A = \sum_{k=0}^{\infty} a_k, A^{'} = \sum_{k=0}^{\infty} |a_k|, A_n = \sum_{k=0}^{n} a_k,$$

$$B = \sum_{k=0}^{\infty} b_k, B^{'} = \sum_{k=0}^{\infty} |b_k|, \quad \text{and} \quad B_n = \sum_{k=0}^{n} b_k.$$

By definition

$$c_k = \sum_{i=0}^{k} a_i b_{k-i}$$

and so

$$\sum_{k=0}^{N} |c_k| \leq \sum_{k=0}^{N} \sum_{i=0}^{k} |a_i| \cdot |b_{k-i}| \leq \left(\sum_{i=0}^{N} |a_i|\right) \left(\sum_{i=0}^{N} |b_i|\right) \leq A^{'} B^{'}.$$

Since the latter two series converge, this provides an upper bound $A^{'}B^{'}$ for the sequence of partial sums $\sum_{k=1}^{N} |c_k|$ and hence the series $\sum_{k=0}^{\infty} c_k$ converges absolutely.

Let us recall that the formal product of the two series is just a particular rearrangement of the terms $a_i b_j$ taken over all $i \geq 0, j \geq 0$. Consider any arrangement of these terms. This must form an absolutely convergent series by the same argument as before since $A^{'}B^{'}$ will be an upper bound for the partial sums of the absolute values $|a_i a_j|$. Thus all rearrangements will converge to the same value.

We can rearrange the terms $a_i b_j$ taken over all $i \geq 0, j \geq 0$ in the following convenient way "by squares." Arrange always so that the first $(m+1)^2 (m = 0, 1, 2, ...)$ terms add up to $A_m B_m$. For example, one such arrangement starts off

$$a_0 b_0 + a_1 b_0 + a_0 b_1 + a_1 b_1 + a_2 b_0 + a_2 b_1 + a_0 b_2 + a_1 b_2 + a_2 b_2 + ...$$

(A picture helps considerably to see the pattern needed.) We know this arrangement converges and we know it must converge to

$$\lim_{m \longrightarrow \infty} A_m B_m = AB.$$

In particular, the series $\sum_{k=0}^{\infty} c_k$ which is just another arrangement, converges to the same number AB as required. ☐

It is possible to improve this theorem to allow one (but not both) of the series to converge nonabsolutely. The conclusion is that the product then converges (perhaps nonabsolutely), but different methods of proof will be needed. As usual, nonabsolutely convergent series are much more fragile, and the free and easy moving about of the terms in this proof is not allowed.

2.8.2 Products of Nonabsolutely Convergent Series

Let us give a famous example, due to Cauchy, of a pair of convergent series whose product diverges. We know that the alternating series

$$\sum_{k=0}^{\infty}(-1)^k \frac{1}{\sqrt{k+1}}$$

is convergent, but not absolutely convergent since the related absolute series is a p-harmonic series with $p = \frac{1}{2}$.

Let

$$\sum_{k=0}^{\infty} c_k$$

be the formal product of this series with itself. By definition the term c_k is given by

$$(-1)^k \left[\frac{1}{\sqrt{1 \cdot (k+1)}} + \frac{1}{\sqrt{2 \cdot (k)}} + \frac{1}{\sqrt{3 \cdot (k-1)}} + ... + \frac{1}{\sqrt{(k+1) \cdot 1}} \right].$$

There are $k+1$ terms in the sum for c_k and each term is larger than $1/(k+1)$ so we see that $|c_k| \geq 1$. Since the terms of the product series $\sum_{k=0}^{\infty} c_k$ do not tend to zero, this is a divergent series.

This example supplies our observation: The formal product of two non absolutely convergent series need not converge. In particular, there may be no convergent series to represent the product

$$\sum_{k=0}^{\infty} a_k \times \sum_{k=0}^{\infty} b_k$$

for a pair of nonabsolutely convergent series. For absolutely convergent series the product always converges.

We should not be too surprised at this result. The theory begins to paint the following picture: Absolutely convergent series can be freely manipulated in most ways and nonabsolutely convergent series can hardly be manipulated in general in any serious manner. Interestingly, the following theorem can be proved that shows that even though, in general, the product might diverge, in cases where it does converge it converges to the "correct" value.

Theorem 2.8.3. ***(Abel)*** *Suppose that* $\sum_{k=0}^{\infty}$ *is the formal product of two nonabsolutely convergent series* $\sum_{k=0}^{\infty} a_k$ *and* $\sum_{k=0}^{\infty} b_k$ *and suppose that this product* $\sum_{k=0}^{\infty} c_k$ *is known to converge. Then*

$$\sum_{k=0}^{\infty} c_k = \sum_{k=0}^{\infty} a_k \times \sum_{k=0}^{\infty} b_k.$$

Proof. The proof requires more technical apparatus and will not be given until Section 2.9.2. □

2.9 Summability Method

A first course in series methods often gives the impression of being obsessed with the issue of convergence or divergence of a series. The huge battery of tests in Section 2.6 devoted to determining the behavior of series might lead one to this conclusion. Accordingly, we may have decided that convergent series are useful and proper tools of analysis while divergent series are useless and without merit.

In fact divergent series are, in many instances, as important or more important than convergent ones. Many eighteenth century mathematicians achieved spectacular results with divergent series but without a proper understanding of what they oure doing. The initial reaction of our founders of nineteenth-century analysis (Cauchy, Abel, and others) was that valid arguments could be based only on convergent series. Divergent series should be shunned. They oure appalled at reasoning such as the following: The series

$$s = 1 - 1 + 1 - 1...$$

can be summed by noting that

$$s = 1 - (1 - 1 + 1 - ...) = 1 - s$$

and so $2s = 1$ or $s = \frac{1}{2}$. But the sum $\frac{1}{2}$ proves to be a useful value for the "sum" of this series even though the series is clearly divergent.

There are many useful ways of doing rigorous work with divergent series. One way, which we now study, is the development of summability methods.

Suppose that a series $\sum_{k=0}^{\infty} a_k$ diverges and yet we wish to assign a "sum" to it by some method. Our standard method thus far is to take the limit of the sequence of partial sums. We write

$$s_n = \sum_{k=0}^{n} a_k$$

and the sum of the series is $\lim_{n\longrightarrow\infty} s_n$. If the series diverges, this means precisely that this sequence does not have a limit. How can we use that sequence or that series nonetheless to assign a different meaning to the sum?

2.9.1 Cesàro's Method

An infinite series $\sum_{k=0}^{\infty} a_k$ has a sum S if the sequence of partial sums

$$s_n = \sum_{k=0}^{n} a_k$$

converges to S. If the sequence of partial sums diverges, then we must assign a sum by a different method. We will still say that the series diverges but, nonetheless, we will be able to find a number that can be considered the sum.

We can replace $\lim_{n\longrightarrow\infty} s_n$, which perhaps does not exist, by

$$\lim_{n\longrightarrow\infty} \frac{s_0 + s_1 + s_2 + \cdots + s_n}{n+1} = C$$

if this exists and use this value for the sum of the series. This is an entirely natural method since it merely takes averages and settles for computing a kind of "average" limit where an actual limit might fail to exist.

For a series $\sum_{k=0}^{\infty} a_k$ often we can use this method to obtain a sum even when the series diverges.

Definition 2.9.1. *If $\{s_n\}$ is the sequence of partial sums of the series $\sum_{k=0}^{\infty} a_k$ and*

$$\lim_{n\longrightarrow\infty} \frac{s_0+s_1+s_2+\cdots+s_n}{n+1} = C$$

then the new sequence

$$\sigma_n = \frac{s_0+s_1+s_2+\cdots+s_n}{n+1}$$

is called the sequence of averages or Cesàro means and we write

$$\sum_{k=0}^{\infty} a_k = C \quad \text{[Cesàro]}.$$

Thus the symbol [Cesàro] indicates that the value is obtained by this method rather than by the usual method of summation (taking limits of partial sums). The method is named after Ernesto Cesàro (1859-1906).

Our first concern in studying a summability method is to determine whether it assigns the "correct" value to a series that already converges. Does

$$\textstyle\sum_{k=0}^{\infty} a_k = A \Longrightarrow \sum_{k=0}^{\infty} a_k = A \text{ [Cesàro]?}$$

Any method of summing a series is said to be regular or a regular summability method if this is the case.

Theorem 2.9.2. *Suppose that a series $\sum_{k=0}^{\infty} a_k$ converges to a value A. Then $\sum_{k=0}^{\infty} a_k = A \quad [Cesro]$ is also true.*

Proof. For any sequence $\{s_n\}$ write

$$\sigma_n = \frac{s_1+s_2+\cdots+s_n}{n}.$$

In that exercise we showed that

$$\lim_{n\longrightarrow\infty} \inf s_n \le \lim_{n\longrightarrow\infty} \inf \sigma_n \le \lim_{n\longrightarrow\infty} \sup \sigma_n \le \lim_{n\longrightarrow\infty} \sup s_n.$$

If we skipped that exercise, here is how to prove it. Let

$$\beta > \lim_{n\longrightarrow\infty} \sup s_n.$$

(If there is no such β, then $\lim_{n\longrightarrow\infty} \sup s_n = \infty$ and there is nothing to prove.) Then $s_n < \beta$ for all $n \ge N$ for some N. Thus

$$\sigma_n \le \frac{1}{n}(s_1+s_2+\cdots+s_{N+1}) + \frac{(n-N+1)\beta}{n}$$

for all $n \ge N$. Fix N, allow $n \longrightarrow \infty$, and take limit superiors of each side to obtain

$$\lim_{n\longrightarrow\infty} \sup \sigma_n \le \beta.$$

It follows that

$$\lim_{n\longrightarrow\infty} \sup \sigma_n \le \lim_{n\longrightarrow\infty} \sup s_n.$$

The other inequality is similar. In particular, if $\lim_{n\longrightarrow\infty} s_n$ exists so too does $\lim_{n\longrightarrow\infty} \sigma_n$ and they are equal, proving the theorem. ☐

Example 2.9.3. *As an example let us sum the series*

$$1 - 1 + 1 - 1 + 1 - 1....$$

The partial sums form the sequence $1, 0, 1, 0, ...$, *which evidently diverges. Indeed the series diverges merely by the trivial test: The terms do not tend to zero. Can we sum this series by the Cesàro summability method? The averages of the sequence of partial sums is clearly tending to* $\frac{1}{2}$. *Thus we can write*

$$\sum_{k=0}^{\infty} (-1)^k = \frac{1}{2} \quad \text{[Cesàro]}$$

even though the series is divergent.

2.9.2 Abel's Method

We require in this section that we recall some calculus limits. We shall need to compute a limit

$$\lim_{x \longrightarrow 1-} F(x)$$

for a function F defined on $(0, 1)$ where the expression $x \longrightarrow 1-$ indicates a left-hand limit. We present a full account of such limits; here we need remember only what this means and how it is computed.

Suppose that a series $\sum_{k=0}^{\infty} a_k$ diverges and yet we wish to assign a "sum" to it by some other method. If the terms of the series do not get too large, then the series

$$F(x) = \sum_{k=0}^{\infty} a_k x^k$$

will converge (by the ratio test) for all $0 \leq x < 1$. The value we wish for the sum of the series would appear to be $F(1)$, but for a divergent series inserting the value 1 for x gives us nothing we can use. Instead we compute

$$\lim_{x \longrightarrow 1-} F(x) = \lim_{x \longrightarrow 1-} \sum_{k=0}^{\infty} a_k x^k = A$$

and use this value for the sum of the series.

Definition 2.9.4. *We write*

$$\sum_{k=0}^{\infty} a_k x^k = A \quad \text{[Abell]}$$

if

$$\lim_{x \longrightarrow 1-} \sum_{k=0}^{\infty} a_k x^k = A.$$

Here the symbol [Abel] indicates that the value is obtained by this method rather than by the usual method of summation (taking limits of partial sums).

As before, our first concern in studying a summability method is to determine whether it assigns the "correct" value to a series that already converges. Does

$$\sum_{k=0}^{\infty} a_k = A \Longrightarrow \sum_{k=0}^{\infty} a_k = A \quad \text{[Abel]?}$$

We are asking, in more correct language, whether Abel's method of summability of series is regular.

Theorem 2.9.5. ***(Abel)*** *Suppose that a series* $\sum_{k=0}^{\infty} a_k$ *converges to a value* A*. Then*

$$\lim_{x \longrightarrow 1-} \sum_{k=0}^{\infty} a_k x^k = A.$$

Proof. Our first step is to note that the convergence of the series $\sum_{k=0}^{\infty} a_k$ requires that the terms $a_k \longrightarrow 0$. In particular, the terms are bounded and so the root test will prove that the series $\sum_{k=0}^{\infty} a_k x^k$ converges absolutely for all $|x| < 1$ at least. Thus we can define

$$F(x) = \sum_{k=0}^{\infty} a_k x^k$$

for $0 \leq x < 1$.

Let us form the product of the series for $F(x)$ with the geometric series

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + ...$$

Since both series are absolutely convergent for any $0 \leq x < 1$, we obtain

$$\frac{F(x)}{1-x} = \sum_{k=0}^{\infty} (a_0 + a_1 + a_2 + \cdots + a_k) x^k.$$

Writing

$$s_k = (a_0 + a_1 + a_2 + \cdots + a_k)$$

and using the fact that

$$s_k \longrightarrow A = \sum_{k=0}^{\infty} a_k,$$

we obtain

$$F(x) = (1-x) \sum_{k=0}^{\infty} s_k x^k = A - (1-x) \sum_{k=0}^{\infty} (s_k - A) x^k.$$

Let $\epsilon > 0$ and choose N so large that

$$|s_k A| < \epsilon/2$$

for $k > N$. Then the inequality

$$|F(x) - A| \leq (1-x) \sum_{k=0}^{N} |s_k - A| x^k + \epsilon/2$$

holds for all $0 \leq x < 1$. The sum here is just a finite sum, and taking limits in finite sums is routine:

$$\lim_{x \longrightarrow 1-} (1-x) \sum_{k=0}^{N} (s_k - A) x^k = 0.$$

Thus for $x < 1$ but sufficiently close to 1 we can make this smaller than $\epsilon/2$ and conclude that

$$|F(x) - A| < \epsilon.$$

We have proved that

$$\lim_{x \longrightarrow 1-} F(x) = A$$

and the theorem is proved. □

Example 2.9.6. *Let us sum the series*

$$\sum_{k=0}^{\infty}(-1)^k = 1 - 1 + 1 - 1 + 1 - 1...$$

by Abel's method. We form

$$F(x) = \sum_{k=0}^{\infty}(-1)^k x^k = \frac{1}{1+x}$$

obtaining the formula by recognizing this as a geometric series. Since

$$\lim_{x \longrightarrow 1-} F(x) = \frac{1}{2}$$

we have proved that

$$\sum_{k=0}^{\infty}(-1)^k = \frac{1}{2} \quad \textit{[Abel]}.$$

Recall that we have already obtained that

$$\sum_{k=0}^{\infty}(-1)^k = \frac{1}{2} \quad \textit{[Cesàro]}$$

so these two different methods have assigned the same sum to this divergent series. we might wish to explore whether the same thing will happen with all series.

Theorem 2.9.7 ((Abel))**.** *Suppose that $\sum_{k=0}^{\infty} c_k$ is the formal product of two convergent series $\sum_{k=0}^{\infty} a_k$ and $\sum_{k=0}^{\infty} b_k$ and suppose that $\sum_{k=0}^{\infty} c_k$ is known to converge. Then*

$$\sum_{k=0}^{\infty} c_k = \sum_{k=0}^{\infty} a_k \times \sum_{k=0}^{\infty} b_k.$$

Proof. The proof just follows on taking limits as $x \longrightarrow 1-$ in the expression

$$\sum_{k=0}^{\infty} c_k x^k = \sum_{k=0}^{\infty} a_k x^k \times \sum_{k=0}^{\infty} b_k x^k.$$

Abel's theorem, allows us to do this. How do we know, however, that this identity is true for all $0 \leq x < 1$? All three of these series are absolutely convergent for $|x| < 1$ and, absolutely convergent series can be multiplied in this way. □

2.10 More on Infinite Sums

How should we form the sum of a double sequence $\{a_{jk}\}$ where both j and k can range over all natural numbers? In many applications of analysis such sums are needed. A variety of methods come to mind:

1. We might simply form the unordered sum

$$\sum_{(j,k)\in\mathbb{N}\times\mathbb{N}} a_{jk}.$$

2. We could construct "partial sums" in some systematic method and take limits just as we do for ordinary series:
$$\lim_{N\longrightarrow\infty}\sum_{j=1}^{N}\sum_{k=1}^{N}a_{jk}.$$
These are called square sums and are quite popular. If we sketch a picture of the set of points
$$\{(j,k):1\le j\le N,1\le k\le N\}$$
in the plane the square will be plainly visible.

3. We could construct partial sums using rectangular sums:
$$\lim_{M,N\longrightarrow\infty}\sum_{j=1}^{M}\sum_{k=1}^{N}a_{jk}.$$
Here the limit is a double limit, requiring both M and N to get large.
If we sketch a picture of the set of points
$$\{(j,k):1\le j\le M,1\le k\le N\}$$
in the plane we will see the rectangle.

4. We could construct partial sums using circular sums:
$$\lim_{R\longrightarrow\infty}\sum_{j^2+k^2\le R^2}a_{jk}.$$
Once again, a sketch would show the circles.

5. We could "iterate" the sums, by summing first over j and then over k:
$$\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}a_{jk}$$
or, in the reverse order,
$$\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}a_{jk}.$$

Our experience in the study of ordinary series suggests that all these methods should produce the same sum if the numbers summed are all nonnegative, but that subtle differences are likely to emerge if we are required to add numbers both positive and negative.

In the exercises there are a number of problems that can be pursued to give a flavor for this kind of theory. At this stage in our studies it is important to grasp the fact that such questions arise. Later, when we have found a need to use these kinds of sums, we can develop the needed theory. The tools for developing that theory are just those that we have studied so far in this work.

2.11 Infinite Products

[1]In this work we studied, quite extensively, infinite sums. There is a similar theory for infinite products, a theory that has much in common with the theory of infinite sums. In this section we shall briefly give an account of this theory, partly to give a contrast and partly to introduce this important topic.

Similar to the notion of an infinite sum

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + a_4 + ...$$

is the notion of an infinite product

$$\prod_{n=1}^{\infty} p_n = p_1 \times p_2 \times p_3 \times p_4 \times ...$$

with a nearly identical definition. Corresponding to the concept of "partial sums" for the former will be the notion of "partial products" for the latter.

The main application of infinite series is that of series representations of functions. The main application of infinite products is exactly the same. Thus, for example, in more advanced material we will find a representation of the sin function as an infinite series

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7...$$

and also as an infinite product

$$\sin x = \left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{4\pi^2}\right)\left(1 - \frac{x^2}{9\pi^2}\right)\left(1 - \frac{x^2}{16\pi^2}\right)....$$

The most obvious starting point for our theory would be to define an infinite product as the limit of the sequence of partial products in exactly the same way that an infinite sum is defined as the limit of the sequence of partial sums. But products behave differently from sums in one important regard: The number zero plays a peculiar role. This is why the definition we now give is slightly different than a first guess might suggest. Our goal is to define an infinite product in such a way that a product can be zero only if one of the factors is zero (just like the situation for finite products).

Definition 2.11.1. *Let $\{b_k\}$ be a sequence of real numbers. We say that the infinite product*

$$\prod_{k=1}^{\infty} b_k$$

converges if there is an integer N so that all $b_k \neq 0$ for $k > N$ and if

$$\lim_{M \longrightarrow \infty} \prod_{k=N+1}^{M} b_k$$

exists and is not zero. For the value of the infinite product we take

$$\prod_{k=1}^{\infty} b_k = b_1 \times b_2 \times ...b_N \times \lim_{M \longrightarrow \infty} \prod_{k=N+1}^{M} b_k.$$

This definition guarantees us that a product of factors can be zero if and only if one of the factors is zero. This is the case for finite products, and we are reluctant to lose this.

Theorem 2.11.2. *A convergent product*

$$\prod_{k=1}^{\infty} b_k = 0$$

if and only if one of the factors is zero.

Proof. This is built into the definition and is one of its features. □

We expect the theory of infinite products to evolve much like the theory of infinite series. We recall that a series $\sum_{k=1}^{n} a_k$ could converge only if $a_k \longrightarrow 0$. Naturally, the product analog requires the terms to tend to 1.

Theorem 2.11.3. *A product*

$$\prod_{k=1}^{\infty} b_k$$

that converges necessarily has $b_k \longrightarrow 1$ *as* $k \longrightarrow \infty$.

Proof. This again is a feature of the definition, which would not be possible if we had not handled the zeros in this way. Choose N so that none of the factors b_k is zero for $k > N$. Then

$$b_n = \lim_{n \longrightarrow \infty} \frac{\prod_{k=N+1}^{n} b_k}{\prod_{k=N+1}^{n-1} b_k} = 1$$

as required. □

As a result of this theorem it is conventional to write all infinite products in the special form

$$\prod_{k=1}^{\infty} (1 + a_k)$$

and remember that the terms $a_k \longrightarrow 0$ as $k \longrightarrow \infty$ in a convergent product. Also, our assumption about the zeros allows for $a_k = -1$ only for finitely many values of k. The expressions $(1 + a_k)$ are called the "factors" of the product and the a_k themselves are called the "terms."

A close linkage with series arises because the two objects

$$\sum_{k=1}^{\infty} a_k \quad \text{and} \quad \prod_{k=1}^{\infty} (1 + a_k),$$

the series and the product, have much the same kind of behavior.

Theorem 2.11.4. *A product*

$$\prod_{k=1}^{\infty} (1 + a_k)$$

where all the terms a_k *are positive is convergent if and only if the series* $\sum_{k=1}^{\infty} a_k$ *converges.*

Proof. Here we use our usual criterion that has served us through most of this work: A sequence that is monotonic is convergent if and only if it is bounded.

Note that

$$a_1 + a_2 + a_3 + \cdots + a_n \leq (1 + a_1)(1 + a_2)(1 + a_3) \times \cdots \times (1 + a_n)$$

so that the convergence of the product gives an upper bound for the partial sums of the series. It follows that if the product converges so must the series.

In the other direction we have

$$(1 + a_1)(1 + a_2)(1 + a_3) \times \cdots \times (1 + a_n) \leq e^{a_1+a_2+a_3+\cdots+a_n}$$

and so the convergence of the series gives an upper bound for the partial products of the infinite product. It follows that if the series converges, so must the product. □

3
Thesis Summary and Future Work

3.1 Thesis Summary

In mathematics, real analysis is the branch of mathematical analysis that studies the behavior of real numbers, sequences and series of real numbers, and real functions. And in this work we have studied one of them in detail, namely infinite sums.

We explored almost all parts of the basic concepts in the infinite sums. Firstly, we began our studies with finite sums that talking about Euler's notation and types of finite sums such as Telescoping Sums, Geometric Progressions, and Summation By Parts. Secondly, we studied infinite unordered sums, especially, about Cauchy criterion. And next section, we studied diverse properties of series and also some special series. After that we studied important criterion for convergence such as boundedness criterion, Cauchy criterion, and absolute convergence.

Furthermore, we studied tests for convergence and it is a special section in this work. We tried to include many remarkable tests to check whether series will be convergence or not. And these are fourteen tests as we have studied so for.

Additionally, we further studied rearrangements such as unconditional convergence, conditional convergence, and comparison of $\sum_{i=1}^{\infty} a_i$ and $\sum_{i\in\mathbb{N}} a_i$. In times, we keep doing about products of series that divide two subsections such as products of absolutely convergent series and products of nonabsolutely convergent series.

Finally, we discussed summability method (Cesàro's method and Abel's method). We also would like to know more about infinite sums and infinite products.

3.2 Future Work

There are many more interesting parts I am passionate about learning in mathematical analysis. In the future, I am hoping to experience continuous functions, differentiations, and integrals and trying to find some connections between them.

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